ON BRUSS' STOPPING PROBLEM WITH GENERAL GAIN FUNCTION

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1. Introduction and summary

The standard secretary problem and its versions was solved by Gilbert and Mosteller [8]. Extension of the secretary problem to a problem of selection of nonextremal candidates has been given by Rose [12], Móri [10] and Szajowski [19]. Other generalizations were presented by Ferguson [7], Sakaguchi [13, 14, 15] and Samuels [16]. Dynkin and Yushkevich [6] and Shiryaev [17] have formulated the secretary problem as an optimal stopping of a Markov sequence. Version of the issue for a random number of object in a finite, continuous time interval was investigated by Cowan and Zabczyk [5]. Another approach to the problem with the random number of object was presented by Presman and Sonin [11]. Bruss [4] extended the model by using a compound Poisson process. Multiple selection for Presman and Sonin version of the problem was investigated by Ano [2]. Ano [1] analyzed the model with Gamma prior distribution. Ano and Ando [3] were considering an extension of the issue to the case with random availability.

Bruss [4] studied a continuous-time generalization of the secretary problem: A man has been allowed a fixed time $T$ to find an apartment. Opportunities to inspect apartments occur at the epochs of a homogeneous Poisson process of unknown intensity $\lambda$. He inspects each apartment immediately when the opportunity arises, and he must decide directly whether to accept or not. At any epoch he is able to rank a given apartment amongst all those inspected to date, where all permutations of ranks are equally likely and independent of the Poisson process. The objective is to maximize the probability of selecting the best apartment from those (if any) available in the interval $[0, T]$.

In this paper we investigate optimal strategies for two versions of Bruss' problem. In these modifications we use different goal functions. In Section 2 we analyze a case when the objective is stop on the best or on the second best object (apartment). In Section 3 the aim is to stop on the second best object. We give the optimal strategies and the value (probability of success) of the optimal strategy for both cases.

Relations of the asymptotic solutions (when the number of objects tends to infinity) of the non-standard secretary problem (see [18] and [19]) to the problem being presented are considered too.
2. Solution of the problem of stopping on the best or on the second best

Let \((j, s)\) denote the state of the process, when the option number \(j\) arrives at time \(s\). Define the relative rank of the \(j\)-th option by \(Y_j\) and its absolute rank by \(X_j\) (for the details see [18]). Based on observation of the relative ranks and the moments of arrivals of the candidates the aim is to stop on the best or on the second best.

Let \(\mathcal{F}_t = \sigma\{N_t, Y_1, Y_2, \ldots, Y_{n_t}\}\) and let \(\mathcal{M}\) be the set of all stopping times with respect to \(\sigma\)-fields \(\{\mathcal{F}_t\}_{t \geq 0}\).

\[
P(X_{\cdot} \in \{1, 2\}) = \sup_{\tau \in \mathcal{M}} P(X_\tau \in \{1, 2\})
\]

One can consider the arrival times only and \(\mathcal{F}_t = \sigma\{S_1, \ldots, S_n, Y_1, \ldots, Y_n\}\), because \(\mathcal{F}_t\) for \(S_n \leq t < S_{n+1}\) is equivalent to \(\mathcal{F}_n\). We can consider equivalently

\[
P(X_{\cdot} \in \{1, 2\}) = \sup_{\tau \in \mathcal{M}^*} P(X_\tau \in \{1, 2\}),
\]

where \(\mathcal{M}^*\) is the set of stopping times with respect to \(\{\mathcal{F}_n\}_{n=1}^\infty\).

Let \(E = \{1, \ldots, \infty\} \times [0, T] \times \{1, 2\}\). For further consideration define process \(\xi_\tau : \Omega \rightarrow E\) such that \(\xi_\tau = (j, S_j, Y_j)\) for some \(j\). Let us define the maximal probability of realizing the goal (stopping on the object with absolute rank equal to 1 or 2) starting from the state \((j, s, r)\):

\[
W^*_j(s) = \sup_{\{\tau \in \mathcal{M}^*: \tau \geq j\}} P(X_\tau \in \{1, 2\} | S_j = s, Y_j = r).
\]

When we are stopping on the object with the relative rank 1 or 2 the payoff is given by formula

\[
g(j, s, r) = U^*_j(s) = \sum_{n=j}^{\infty} P(X_n \in \{1, 2\}, N(T) = n \mid S_j = s, Y_j = r)
\]

\[
= \left\{ \begin{array}{ll}
\sum_{n=j}^{\infty} \frac{j(2n-j-1)}{n(n-1)} P(N(T) = n \mid S_j = s) & \text{for } r = 1, \\
\sum_{n=j}^{\infty} \frac{j(j+1)}{n(n-1)} P(N(T) = n \mid S_j = s) & \text{for } r = 2.
\end{array} \right.
\]

Well known are the following two facts

**Fact 1** (see [9] and [14]) Let \(S_1, S_2, \ldots\) denote the arrival times of the Poisson process \(\{N_t\}_{t \geq 0}\). For unknown intensity \(\lambda\) an exponential prior density \(g(\lambda) = ae^{-a\lambda} \mathbb{1}_{\{\lambda > 0\}}(\lambda)\) is assumed, where \(a\) is known, positive parameter. By Bayes' theorem, the conditional posterior density is of the form

\[
f(\lambda \mid S_j = s) = f(\lambda \mid S_j = s, S_{j-1} = s_{j-1}, \ldots, S_1 = s_1)
\]

\[
= \frac{\lambda^j}{j!} (s + a)^{j+1} e^{-(s+a)} \mathbb{1}_{\{\lambda > 0\}}(\lambda), s \in [0, T].
\]

**Fact 2** (see [14]) The posterior distribution of the number of objects \(N(t)\) given moments of arrival is a Pascal distribution and:

\[
P(N(T) = n \mid S_1 = t_1, \ldots, S_{j-1} = t_{j-1}, S_j = s) = \frac{n!}{j!(n-j)!} \left( \frac{s + a}{T + a} \right)^{j+1} \left( 1 - \frac{s + a}{T + a} \right)^{n-j}.
\]
For \(0 < p < 1\) and \(q = 1 - p\) we have the following usefull formula:

\[
\sum_{k=0}^{\infty} \binom{r + k - 1}{k} p^r q^k = p^r \frac{1}{(1 - q)^r} = 1,
\]

(2)

\[
\sum_{k=0}^{\infty} k \binom{r + k - 1}{k} p^r q^k = r \frac{q}{p}.
\]

(3)

Using (for the details see [18])

\[
P(X_j \in \{1, 2\}, N(T) = n | Y_j = 1) = \frac{j(2n - j - 1)}{n(n - 1)},
\]

\[
P(X_j \in \{1, 2\}, N(T) = n | Y_j = 2) = \frac{j(j - 1)}{n(n - 1)}
\]

and (3) combined with formula given by Fact 2 we calculate \(U_j^r(s)\) for \(r = 1\)

\[
U_j^1(s) = \sum_{n=j}^{\infty} \frac{j(2n - j - 1)}{n(n - 1)} \binom{n}{j} \frac{(s + a)^{j+1}}{T + a} \left(1 - \frac{s + a}{T + a}\right)^{n-j}
\]

\[
= \sum_{n=j}^{\infty} \frac{j(2n - j - 1)}{n(n - 1)} \frac{(s + a)^{j+1}}{T + a} \sum_{r=0}^{\infty} \binom{r + j - 2}{r} \frac{r + j + r + 1}{j - 1} \left(1 - \frac{s + a}{T + a}\right)^r
\]

\[
= \frac{(s + a)}{T + a} \left\{ \sum_{r=0}^{\infty} \frac{r}{j - 1} \binom{r + (j - 1) - 1}{r} \left(1 - \frac{s + a}{T + a}\right)^r \right\} + \sum_{r=0}^{\infty} \frac{r + (j - 1) - 1}{j - 1} \left(1 - \frac{s + a}{T + a}\right)^r
\]

\[
= \frac{(s + a)}{T + a} \left\{ \sum_{r=0}^{\infty} \frac{r}{j - 1} \binom{r + (j - 1) - 1}{r} \left(1 - \frac{s + a}{T + a}\right)^r \right\} + \sum_{r=0}^{\infty} \frac{(r + (j - 1) - 1)}{j - 1} \left(1 - \frac{s + a}{T + a}\right)^r
\]

\[
= \frac{(s + a)}{T + a} \left(2 - \frac{s + a}{T + a}\right).
\]

and in the similar manner for \(r = 2\)

\[
U_j^2(s) = \sum_{n=j}^{\infty} \frac{j(j - 1)}{n(n - 1)} \binom{n}{j} \frac{(s + a)^{j+1}}{T + a} \left(1 - \frac{s + a}{T + a}\right)^{n-j}
\]

\[
= \sum_{n=j}^{\infty} \frac{j(j - 1)}{n(n - 1)} \frac{(s + a)^{j+1}}{T + a} \sum_{r=0}^{\infty} \binom{r + j - 2}{r} \left(1 - \frac{s + a}{T + a}\right)^r
\]

\[
= \frac{(s + a)^2}{(T + a)^2}.
\]
Summarizing the results obtained we get

\[ U_j^r(s) = \begin{cases} \frac{s^{r+1}}{r+1} (2 - \frac{s^{r+1}}{r+1}) & \text{for } r = 1, \\ \left(\frac{s^{r+1}}{r+1}\right)^2 & \text{for } r = 2. \end{cases} \]

Define the probability of realizing the goal doing one step more starting from \((j, s, r_j)\) and next proceeding according to the optimal strategy by

\[ V_j^{r_j}(s) = \int_0^{T-s} \sum_{r_{j+k} = 1}^2 \sum_{k=1}^\infty p_{(j, s, r_j)}^{(k,u,r_{j+k})} W_{j+k}^{r_{j+k}}(s + u) du, \]

where the probability of one-step transition from state \((j, s, r_j)\) to state \((j + k, s + u, r_{j+k})\) is given by formula

\[ p_{(j, s, r_j)}^{(k,u,r_{j+k})} = \int_0^\infty \mathbf{P}(S_{j+k} = s + u \mid S_j = s, \lambda) \times \mathbf{P}(Y_{j+k} = r_{j+k} \mid Y_j = r_j, S_j = s, S_{j+k} = s + u, \lambda) \cdot f(\lambda \mid S_j = s) d\lambda \]

with

\[ \mathbf{P}(Y_{j+k} = r_{j+k} \mid Y_j = r_j, S_j = s, S_{j+k} = s + u, \lambda) = \begin{cases} \frac{j}{(j-k)(j+k-1)} & \text{for } r_j, r_{j+k} = 1, \\ \frac{j}{(j-k)(j+k-1)} & \text{for } r_j, r_{j+k} \in \{1, 2\}. \end{cases} \]

Basing on equality

\[ \int_0^\infty \lambda^{k+j} \exp(-\lambda(s + a + u)) d\lambda = \frac{\Gamma(k + j + 1)}{(s + a + u)^{k+j+1}} \]

we have transition probabilities for \(r_j, r_{j+k} = 1\) (see [1])

\[ p_{(j, s, r_j)}^{(k,u,r_{j+k})} = \int_0^\infty \frac{\lambda e^{-\lambda u} (\lambda u)^{k-1}}{(k-1)!} \frac{j}{(j+k)(j+k-1)} \frac{e^{-\lambda(s+a)}(s+a)^{j+1}}{j!} d\lambda \]

\[ = \frac{s + a}{(s + a + u)^2} \left(\frac{j + k - 2}{k - 1}\right) \left(\frac{s + a}{s + a + u}\right)^j \left(\frac{u}{s + a + u}\right)^{k-1}. \]  

(4)

and for \(r_j, r_{j+k} \in \{1, 2\}\)

\[ p_{(j, s, r_j)}^{(k,u,r_{j+k})} = \int_0^\infty \frac{\lambda e^{-\lambda u} (\lambda u)^{k-1}}{(k-1)!} \frac{j(j-1)}{(j+k)(j+k-1)(j+k-2)} \frac{e^{-\lambda(s+a)}(s+a)^{j+1}}{j!} d\lambda \]

\[ = \frac{s + a}{(s + a + u)^2} \left(\frac{j + k - 3}{k - 1}\right) \left(\frac{s + a}{s + a + u}\right)^j \left(\frac{u}{s + a + u}\right)^{k-1}. \]  

(5)

By the principle of optimality we have the optimal equation:

\[ W_j^r(s) = \max \{ U_j^r(s), V_j^r(s) \} \text{ for } j = 1, 2, \ldots, s \in [0, T], r \in \{1, 2\} \]

Let \(B\) be the optimal stopping set. We will show that this set can be divided into two parts

\[ B = B_{1}(\alpha, \beta) \cup B_{1,2}(\beta, 1), \]

(6)

\[ = \{(j, s, 1) : \alpha \leq s \leq \beta\} \cup \{(j, s, r) : \beta \leq s \leq 1, r = 1, 2\}. \]  

(7)
In the first of two sets defined we stop only on the relatively first objects and in the latter on the relatively first or on the second.

Let ̂B_{1,2} be the one-step look-ahead (OLA) stopping region. It means that ̂B_{1,2} is the set of states (j, s, r) for which selecting the current relatively best or second best option is at least as good as waiting for the next relatively best or second best option to appear and then selecting it. We will prove that OLA stopping set is optimal stopping set. Define the average payoff (see (1) for definition of g(·)) for doing one step more by

\[ R_j(s) = T g(j, s, \cdot) = \int_0^{T-s} \sum_{r,j+k=1}^{2} \sum_{u=1}^{\infty} \sum_{j+k=1}^{\infty} p_{(j,s,r)}^{(k,u,r_j+k)} u_{r_j+k}^r(s + u) du. \]

In other words, \( R_j(s) \) is one-step expected reward and it is obtained using one-step transition operator.

We have the following formula for \( R_j(s) \)

\[
R_j(s) = \int_0^{T-s} \sum_{k=1}^{\infty} \frac{s + a}{(s + a + u)^2} \left( \frac{j + k - 3}{k - 1} \right) \left( \frac{s + a}{s + a + u} \right)^j \left( \frac{u}{s + a + u} \right)^{k-1} \times \left[ \frac{s + a + u}{T + a} \left( 2 - \frac{s + a + u}{T + a} \right) + \left( \frac{s + a + u}{T + a} \right)^2 \right] du
\]

\[
= 2 \int_0^{T-s} \frac{(s + a)^2}{(s + a + u)^2} \frac{1}{T + a} \left( 1 - \frac{s + a}{T + a} \right) du = \frac{s + a}{T + a} \left( 1 - \frac{s + a}{T + a} \right)
\]

Therefore the set ̂B_{1,2} is given by formula

\[ ̂B_{1,2} = \{(j, s, r) : U^*_j(s) - R_j(s) \geq 0, r = 1, 2\} \]

In order to find \( \beta \), we are solving these inequalities. Let us define

\[
h_j^1(s) = U^*_j(s) - R_j(s) = \frac{s + a}{T + a} \left( 2 - \frac{s + a}{T + a} \right) - \frac{2(s + a)}{T + a} \left( 1 - \frac{s + a}{T + a} \right)
\]

\[
= \left( \frac{s + a}{T + a} \right)^2
\]

and

\[
h_j^2(s) = U^*_j(s) - R_j(s) = \left( \frac{s + a}{T + a} \right)^2 - \frac{2(s + a)}{T + a} \left( 1 - \frac{s + a}{T + a} \right)
\]

\[
= 3 \left( \frac{s + a}{T + a} \right)^2 - 2 \frac{s + a}{T + a}
\]

In the calculations above we were using (2).

Because \( h_j^1(s) \geq 0 \) holds for any \( s \in [0, T] \), \( \beta \) is the nearest point on the left hand side of 1 at which \( h_j^2(s) \geq 0 \) does not hold.

Let us introduce a linear time transformation \( \hat{\phi}(s) = \frac{s + r}{T + a} \). The optimal strategies will be given using this function.
Then we have
\[
\tilde{B}_{1,2}(\beta, 1) = \{(j, s, t) : t \in \{1, 2\}, h_j^*(s) \geq 0\} = \left\{(j, s, t) : t \in \{1, 2\}, \frac{s + a}{T + a} \geq \frac{2}{3}\right\}
\]
\[
= \{(j, s, t) : t \in \{1, 2\}, s \geq \beta\},
\]
where
\[
\beta = t^* = \min\left\{s : \frac{s + a}{T + a} \geq \frac{2}{3}\right\} = \phi^{-1}(2/3).
\]  (8)

We will prove that above defined $B_{1,2}(\beta, 1)$ forms a part of the optimal stopping set, moreover we will prove that the optimal stopping set has form (6).

It is clear that $R_{1}(t) = T\tilde{U}^*_j(t) = Tg(j, t, r) = \mathbf{E}_{\{j, t, r\}}g(\xi_1)$. Let $Q\tilde{U}^*_j(t) = \max\{U^*_j(t), T\tilde{U}^*_j(t)\}$. Payoff is defined by $W^*_j(t) = \lim_{n \to \infty} Q^n\tilde{U}^*_j(t)$. OLA stopping set is defined as $\tilde{B}_{1,2} = \{(j, t, r) : U^*_j(t) - T\tilde{U}^*_j(t) \geq 0, r = 1\}$. While the optimal stopping set is given by $B = \{(j, t, r) : U^*_j(t) - TW^*_j(s) \geq 0, r = 1\}$. It is known that $B(j, t, r) \subseteq B(j, s, r)$ for $s > t > t^* = \beta$.

Let $(j, t, r) \in \tilde{B}_{1,2}$. From definition of $B_{1,2}$ we get $U^*_j(t) - T\tilde{U}^*_j(t) \geq 0$. It implies $Q\tilde{U}^*_j(t) = U^*_j(t)$, hence $W^*_j(t) = \lim_{n \to \infty} Q^n\tilde{U}^*_j(t) = U^*_j(t)$ what yields $(j, t, r) \in B$. OLA stopping set $\tilde{B}_{1,2}$ is part $B_{1,2}$ of the optimal stopping set $B$.

Take $t < t^*$. Combining $U^*_j(t) < T\tilde{U}^*_j(t)$ and inequality $T\tilde{U}^*_j(t) < W^*_j(t)$ derived from definitions of the operators $T$ and $Q$ we get immediately $U^*_j(t) < W^*_j(t)$, what implies that $(j, t, 2)$ does not belong to $B$. It means that there is no need to stop at $r = 2$ below $t^*$.

$U^*_j(t)$ is decreasing for $t \leq t^*$, $W^*_j(t)$ is nonincreasing and both functions are continuous. Basing on these and using $U^*_j(t^*) = W^*_j(t^*) > T\tilde{U}^*_j(t^*)$ there exists $\epsilon > 0$ such that for any $t \in (t^* - \epsilon, t^*)$ we have $W^*_j(t) > T\tilde{U}^*_j(t)$, what implies $(j, r, 1) \in B$. Concluding, we have to stop at $r = 1$ for some $t < t^*$.

The results above imply that the optimal stopping set $B$ has the form given by (6).

Let us define a family of sets $D_t$ for $t \in [0, T]$, $D_t = \{(j, s, 1) : s > t\} \cup \{(j, s, 2) : s > t\}$ and let $\tau = \inf\{t : \xi_t \in D_t\}$ be the moment of the first reaching of the set $D_t$. Denote $\xi_{\tau}$ the new process. Let $t \in (t^* - \epsilon, t^*)$ and $t < s < t^*$. We calculate expectation with respect to the new process using probabilities of reaching sets $D_s$

\[
\mathbf{E}_{(j, s, 2)}g(\xi_{\tau}) = \int_{0}^{s-t} \sum_{k=1}^{\infty} \frac{s + a}{(s + a + u)^2} \left(\frac{j + k - 2}{k - 1}\right) \left(\frac{s + a}{s + a + u}\right)^j \left(\frac{u}{s + a + u}\right)^{k-1}
\]
\[
\times \left(\frac{s + a + u}{T + a} \left(2 - \frac{s + a + u}{T + a}\right)ight) du
\]
\[
+ \left(1 - \int_{0}^{s-t} \sum_{k=1}^{\infty} \frac{s + a}{(s + a + u)^2} \left(\frac{j + k - 2}{k - 1}\right) \left(\frac{s + a}{s + a + u}\right)^j \left(\frac{u}{s + a + u}\right)^{k-1}
\]
\[
\times \int_{0}^{T-s} \sum_{k=1}^{\infty} \frac{s + a}{(s + a + u)^2} \left(\frac{j + k - 3}{k - 1}\right) \left(\frac{s + a}{s + a + u}\right)^j \left(\frac{u}{s + a + u}\right)^{k-1}
\]
\[
\right)
\]  (9)
\[
\times \int_{0}^{T-s} \sum_{k=1}^{\infty} \frac{s + a}{(s + a + u)^2} \left(\frac{j + k - 3}{k - 1}\right) \left(\frac{s + a}{s + a + u}\right)^j \left(\frac{u}{s + a + u}\right)^{k-1}
\]
\[
\right)
\]  (10)
\[ \left( \frac{s + a + u}{T + a} \left( 2 - \frac{s + a + u}{T + a} \right) + \left( \frac{s + a + u}{T + a} \right)^2 \right) du \]

\[ = 2 \int_0^{s^*} \frac{s + a}{T + a} \frac{1}{s + a + u} du \]

\[ - \int_0^{s^*} \left( \frac{s + a}{T + a} \right)^2 du + \left[ 1 - \frac{s + a}{s^* + a} \right] \cdot \frac{s^* + a}{T + a} \left( 1 - \frac{s^* + a}{T + a} \right) \]

\[ = 2 \frac{s + a}{T + a} \ln \frac{s^* + a}{s + a} - \frac{s + a}{T + a} \left( s^* + a - (s + a) \right) + 2 \frac{s + a}{s^* + a} \frac{s^* + a}{T + a} \left( 1 - \frac{s^* + a}{T + a} \right), \]

where (9) is probability that the relatively first object has not appeared in an interval \([s, t^*]\) and (10) is the expected value of the optimal procedure at \((j, t^*, r)\) for any \(j\), fixed \(t^*\) defined by (8) and \(r \in \{1, 2\}\).

Because \(B_{1,2}\) is a part of the optimal stopping set we have \(E_{(j,t_1)}g(S_{r_1}) = W_j^1(t)\).

Function \(U_j^1(t)\) is monotone, therefore exists a unique \(s^*\) such that \(U_j^1(t) < W_j^1(t)\) for \(t < s^*\) and \(U_j^1(t) \geq W_j^1(t)\) for \(t \geq s^*\).

Set \(B(\alpha, \beta) = D^*, \alpha = s^*, \beta = t^*\) and \(s^*\) is solution of the equation:

\[ h_j^1(s) = U_j^1(s) - \psi_j^1(s) = \frac{s + a}{T + a} \left( 2 - \frac{s + a}{T + a} \right) - 2 \frac{s + a}{T + a} \ln \frac{s^* + a}{s + a} \]

\[ + \frac{s + a}{(T + a)^2} \left( s^* + a - (s + a) \right) - 2 \frac{s + a}{s^* + a} \frac{s^* + a}{T + a} \left( 1 - \frac{s^* + a}{T + a} \right), \]

For \(x = \frac{s + a}{T + a} = \phi(s)\) and \(y = \frac{s^* + a}{T + a} = \phi(s^*)\) we get that \(h_j^1(s)\) takes the form

\[ h_j^1(s) = x(2 - x) - 2x \ln \frac{y}{x} + x(y - x) - 2x(1 - y). \tag{11} \]

From the definition of \(t^*\) (see (3)) we have \(y = \phi(\beta) = 2/3\). Solving the equation \(h_j^1(s) = 0\) we get \(x = \phi(\alpha) \approx 0.347\). The value of the problem is \(v \approx 0.573\).

3. Solution of the problem of stopping on the second best

This part of the paper is devoted to the optimal selection of the second best candidate. We are working using the framework presented in Section 2. As in Section mentioned, we have

\[ W_j^r(s) = \sup_{\{r \in \mathbb{N}^*: r \geq j\}} P(X_j = 2 \mid S_j = s, Y_j = r), \]

\[ P(X_j = 2, N(T) = n \mid Y_j = r) = \begin{cases} \frac{j(n-j)}{n(n-1)} & \text{for } r = 1, \\ \frac{j(n-j)}{n(n-1)} & \text{for } r = 2, \end{cases} \]

Using (2) and (3) in calculations as in Section 2 we get

\[ U_j^r(s) = \sum_{n=j}^{\infty} P(X_j = 2, N(T) = n \mid S_j = s, Y_j = r) \]
\[ R_j(s) = \int_0^{T-s} \sum_{r_j \leq k \leq 2} \sum_{n \geq j} p_{(n,s)}^{(r_j,s+k)} U_{j+r_j+k}^s(s+u) du. \]

Using transition probabilities given by (4) and (5) we get

\[ R_j(s) = \int_0^{T-s} \sum_{k=1}^{\infty} \frac{s + a}{(s + a + u)^2} \left( \frac{j + k - 3}{k - 1} \right) \left( \frac{s + a}{s + a + u} \right)^j \left( \frac{u}{s + a + u} \right)^{k-1} \left[ \frac{s + a + u}{T + a} \left( 1 - \frac{s + a + u}{T + a} \right) + \left( \frac{s + a + u}{T + a} \right)^2 \right] du \]

\[ = \frac{s + a}{T + a} \left( 1 - \frac{s + a}{T + a} \right). \]

We can show using argumentation as in Section 2 that this set can be divided into two parts

\[ C = C_1(\alpha, \beta) \cup C_{1,2}(\beta, 1) \]
\[ = \{ (j, s, 1) : \alpha \leq s \leq \beta \} \cup \{ (j, s, r) : \beta \leq s \leq 1, r = 1, 2 \}. \]

Part \( C_{1,2} \) of the set \( C \) is OLA stopping set given by formula (proof as in Section 2)

\[ C_{1,2} = \{ (j, s, r) : U_j^r(s) - R_j(s) \geq 0, r = 1, 2 \}. \]

In order to find the set \( C_{1,2}(\beta, 1) \) we are solving the following inequalities

\[ h_j^1(s) = U_j^1(s) - R_j(s) = \frac{s + a}{T + a} \left( 1 - \frac{s + a}{T + a} \right) - \frac{s + a}{T + a} \left( 1 - \frac{s + a}{T + a} \right) = 0 \]

and

\[ h_j^2(s) = U_j^2(s) - R_j(s) = \left( \frac{s + a}{T + a} \right)^2 - \frac{s + a}{T + a} \left( 1 - \frac{s + a}{T + a} \right) \]
\[ = 2 \left( \frac{s + a}{T + a} \right)^2 - \frac{s + a}{T + a}. \]

In calculations we were using (3).

Because \( h_j^1(s) \geq 0 \) holds for any \( s \in [0, T] \), we will obtain \( \beta \) solving the inequality \( h_j^2(s) \geq 0 \).
Then
\[
C_{1,2}(\beta, 1) = \{(j, s, t) : t \in \{1, 2\}, h_j^2(s) \geq 0\} = \{(j, s, t) : t \in \{1, 2\}, \frac{s + a}{T + a} \geq \frac{1}{2}\} = \{(j, s, t) : t \in \{1, 2\}, s \geq \beta\},
\]
where
\[
\beta = s^* = \min \left\{ s : \frac{s + a}{T + a} \geq 1/2 \right\} = \phi^{-1}(1/2).
\]

In order to find the set \(C_1(\alpha, \beta)\) we define
\[
h_j^1(s) = \frac{s + a}{T + a} \left( 1 - \frac{s + a}{T + a} \right)
- \left\{ \int_0^{s^*} \sum_{k=1}^{\infty} \frac{s + a}{(s + a + u)^2} \left( \frac{s + a}{s + a + u} \right)^j \left( \frac{u}{s + a + u} \right)^{k-1} \cdot \frac{s + a + u}{T + a} \left( 1 - \frac{s + a + u}{T + a} \right) du \right. 
+ \left( 1 - \int_0^{s^*} \sum_{k=1}^{\infty} \frac{s + a}{(s + a + u)^2} \left( \frac{s + a}{s + a + u} \right)^j \left( \frac{u}{s + a + u} \right)^{k-1} du \right)
\times \int_0^{T-s^*} \sum_{k=1}^{\infty} \frac{s + a}{(s + a + u)^2} \left( \frac{s + a}{s + a + u} \right)^j \left( \frac{u}{s + a + u} \right)^{k-1} \cdot \frac{s + a + u}{T + a} \left( 1 - \frac{s + a + u}{T + a} \right) du \}
\]
\[
= x(1 - x) - x \ln \frac{y}{x} + x(y - x) - \frac{x^2}{y x + 1 - y},
\]
where \(x = \frac{s + a}{T + a} = \phi(s)\) and \(y = \frac{s^* + a}{T + a} = \phi(s^*)\).

Solving the equation \(h_j^1(s) = 0\) for \(y = \phi(\beta) = 1/2\) we get \(x = \phi(\alpha) = 1/2\). In fact we have \(C = C_{1,2}(\beta, 1)\). The value of the problem is \(c = 1/4\).

Let us look more closely on the result obtained. Define two average one-step payoffs: the first one when we are stopping on relatively first or second candidate (the standard one) and the latter when we are stopping on the second relatively best. Let us denote them by \(R_j(s)\) and \(R_j(s)\) (for \(s \geq \beta\), respectively. This method gives us optimal stopping set because each set of the form \(\{(j, s, r) : s \geq s^*, r = 1, 2\}\) is a part of the optimal stopping set. We have the following identities
\[
R_j(s) = \int_0^{T-s} \sum_{k=1}^{\infty} \frac{s + a}{(s + a + u)^2} \left( \frac{s + a}{s + a + u} \right)^j \left( \frac{u}{s + a + u} \right)^{k-1} \times \left[ \frac{s + a + u}{T + a} \left( 1 - \frac{s + a + u}{T + a} \right) + \left( \frac{s + a + u}{T + a} \right)^2 \right] du
= \frac{s + a}{T + a} \left( 1 - \frac{s + a}{T + a} \right)
\]
and

\[ \hat{R}_j(s) = \int_0^{T-s} \sum_{k=1}^\infty \frac{s + a}{(s + a + u)^2} \binom{j + k - 2}{k - 1} \left( \frac{s + a}{s + a + u} \right)^j \left( \frac{u}{s + a + u} \right)^{k-1} \left( \frac{s + a + u}{T + a} \right)^2 \, du \]

\[ = \frac{s + a}{T + a} \left( 1 - \frac{s + a}{T + a} \right). \]

We can see that \( R_j(s) \equiv \hat{R}(s) \). It means that stopping on the relatively second best only is as good as stopping on the relatively best or on the second best.

There is possibility to define other optimal strategies

\[ D = D_1(\beta, 1) \cup D_2 = \{(s, 2) : \beta \leq s \leq 1\} \cup D_2, \]

where \( D_2 \) is any subset of \( \{(s, 1) : \beta \leq s \leq 1\} \). In particular \( D_2 \) can be an empty set.

The sets \( C \) and \( D \) describe the different strategies with equal payoffs.

4. Conclusions

We have given the optimal strategies for two non-standard versions of Bruss’ problem. In both problems presented we have threshold strategies. In the problem when our goal is to stop the process on the second best object (Section 3) we have got an interesting result. According to the optimal strategy we have to stop for moments \( \phi(s) \geq \frac{1}{2} \) on the best or on the second best object. Moreover, we do not stop on any object before the moment \( \phi(s) = \frac{1}{2} \). There are more possibilities. We have a wide variety of optimal strategies. For \( \frac{1}{2} \leq \phi(s) \leq 1 \) we should stop on the relatively second, but we are free to decide whether to stop at the relatively best or not.

The strategies obtained for both problems are equivalent (after application of the linear time scaling \( \phi(\cdot) \)) to the asymptotically optimal strategies in adequate versions of the standard no-information secretary problem (CSP). Solution of problem of selecting the second best object can be find in [19]. Problem of selection of object with absolute rank belonging to \{1, 2\} can be solved using method presented in [18]). This relationship between the optimal strategies in two problems is a subject for further research.

REFERENCES


