THREE-PERSON STOPPING GAME WITH PLAYERS HAVING PRIVILEGES

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1. Introduction

In the paper, a construction of Nash equilibria for a random-priority, finite-horizon, three-person, nonzero-sum stopping game is given. Let \((X_n, \mathcal{F}_n, P_n)_{n=0}^N\) be a homogeneous Markov process defined on a probability space \((\Omega, \mathcal{F}, P)\) with state space \((E, \mathcal{B})\). At each moment \(n = 1, 2, \ldots, N\), the decision-makers (henceforth called Player 1, Player 2, and Player 3) are able to observe the Markov chain sequentially. Each player has his own utility function \(g_i : E \rightarrow \mathbb{R}\), \(i = 1, 2, 3\), and at moment \(n\) each decides separately whether to accept or reject the realization \(x_n\) of \(X_n\). We assume that the functions \(g_i\) are measurable and bounded. If it happens that two or three players have selected the same moment \(n\) to accept \(x_n\), then a lottery decides which player gets the right (priority) of acceptance. Let \(0 \leq \gamma_n \leq \delta_n \leq 1\) for \(n = 1, 2, \ldots, N\). According to the lottery, at moment \(\tau\), if three players would like to accept \(x_\tau\), then Player 1 is chosen with probability \(\gamma_\tau\), Player 2 with probability \(\delta_\tau - \gamma_\tau\), and Player 3 with probability \(1 - \delta_\tau\). If only two players compete for the observation \(x_\tau\), then the priority of Player 1 is proportional to \(\gamma_\tau\), Player 2's priority is proportional to \(\delta_\tau - \gamma_\tau\), and Player 3's priority is proportional to \(1 - \delta_\tau\). The players rejected by the lottery may select any other realization \(x_n\) at a later moment \(n\), \(\tau < n \leq N\). Once accepted, a realization cannot be rejected; once rejected, it cannot be reconsidered. If a player has not chosen any realization of the Markov process, he gets \(g_i^* = \inf_{x \in E} g_i(x)\).

The aim of each player is to choose a realization that maximizes his expected utility. The problem will be formulated as a three-person non-zero-sum game with the concept of Nash equilibrium as the solution. The two-person non-zero-sum stopping game with permanent priority for Player 1 has been solved by Ferstenberg [2]. Random priority for such games has been considered by Szajowski [14]. The \(n\)-person stopping-game model has been investigated by Enns and Ferstenberg [1]. Such games are also strictly connected with the optimal stopping of stochastic processes. The ideas of Kuhn [5] and Rieder [9] as well as Yasuda [15] and Ohtsuko [7] are adopted to this random-priority game model. The inspiration for these game models is the secretary problem. For the original secretary problem and its extension, the reader is referred to Gilbert and Mosteller [4], Freeman [3] or Rose [10]. Related games can be found in [6, 8, 11-13], where non-zero-sum versions of the games have been investigated. A review of these problems can be found in [8]. In noncooperative non-zero-sum games, one of the possible definitions of a solution is the Nash equilibrium. This approach is adopted here.

The mathematical model of the problem formulated above will be presented in Sec. 2, and equilibria for each lottery will be derived in Sec. 3.

2. The Game with Random Priority

In problems of optimal stopping, the basic class of strategies \(T^N\) are Markov times with respect to \(\sigma\)-fields \((\mathcal{F}_n)_{n=0}^N\). We permit \(P(\tau \leq N) < 1\) for some \(\tau \in T^N\). This class of strategies is not sufficient in the stopping game (see [15]). So we consider a class of randomized stopping times. It is assumed that the probability space is rich enough to admit the following constructions.

**Definition 1** (see [15]). A simple strategy can be described by a random sequence \(p = (p_n) \in \mathcal{P}^N\) such that for each \(n\): (i) \(p_n, q_n\) and \(r_n\) are adapted to \(\mathcal{F}_n\); (ii) \(0 \leq p_n, q_n, r_n \leq 1\) a.s. If each random variable equals either 0 or 1, we call such a strategy a pure strategy.

Let \(A_1, \ldots, A_N\) be independent identically distributed random variables (i.i.d.r.v.) from the uniform distribution on \([0, 1]\) and independent of the Markov process \((X_n, \mathcal{F}_n, P_n)_{n=0}^N\). Let \(\mathcal{H}_n\) be the \(\sigma\)-field generated by \(\mathcal{F}_n, A_1, \ldots, A_n\). A randomized Markov time \(\tau(p)\) for strategy \(p = (p_n) \in \mathcal{P}^N\) is defined by \(\tau(p^*) = \inf\{N \geq n \geq 1 : A_n \leq p_n^*\}\). We denote by \(\mathcal{M}^N, i = 1, 2, 3\), the sets of all randomized strategies of the \(i\)th player. Clearly, if each \(p_n^*\) is either zero or one, then the strategy is pure and \(\tau(p)\) is, in fact, an \((\mathcal{F}_n)\)-Markov time. In particular, an \((\mathcal{F}_n)\)-Markov time \(\tau^*\) corresponds to the strategy \(p = (p_n^*)\) with \(p_n^* = I(\tau = n)\), where \(I_A\) is the indicator function for the set \(A\).


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In the game, considering the class of randomized Markov times is not adequate, because they allow both players to stop at the same time. Random assignment of priority to a player requires us to consider modified strategies. Denote

$$\mathcal{T}^N = \{ r \in \mathcal{T}^N : \tau_r \geq k \},$$

$$\mathcal{P}^N = \{ p \in \mathcal{P}^N : p_i = 0 \text{ for } i = 1, 2, \ldots, k-1 \}.$$

One can define the sets of strategies $\mathcal{M}^N_i = \{ (p', \xi_i) : p' \in \mathcal{P}^N, \tau_{p'} = \tau_{p'_{n+1}} \text{ for every } n \}$ for Player $i$.

Let $\xi_1, \xi_2, \ldots$ be i.i.d.r.v. uniformly distributed on $[0, 1]$ and independent of $\mathcal{Y}_n$, and the lottery be given by the sequence of divisions of the interval $[0, 1]$. Denote these divisions by $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_N$, where $\mathcal{C}_n = (C_{n1}, C_{n2}, C_{n3})$, $C_{n1} \cap C_{k2} = \emptyset$ for $i \neq j, i, j = 1, 2, 3$, and $C_{n1} \cup C_{n2} \cup C_{n3} = [0, 1]$ for $n = 1, 2, \ldots, N$. The lottery is used in such a way that if $\lambda_i(p') = \lambda_j(p') = n$ and $\lambda_k(p') \neq n$, then $\xi_n \in C_{n1} \cup C_{n2}, i, j, k = 1, 2, 3, i \neq j$. Denote $\mathcal{H}_n = \sigma(H_0, \xi_1, \xi_2, \ldots, \xi_n)$, and let $\mathcal{T}^N$ be the set of Markov times with respect to $\mathcal{H}_n$. For every triple $(s^1, s^2, s^3)$ such that $s^i \in \mathcal{M}^N_i$, we define

$$\tau_i(s^1, s^2, s^3) = \lambda_i(p') I(\lambda_i(p') < \lambda_j(p') \wedge \lambda_k(p'))$$

$$+ \lambda_i(p') I(\xi_{n+1} \in \mathcal{C}_{n}(\lambda_i(p'))) + s_i^{1+2}(s^1, s^2, s^3) I(\xi_{n+1} \in \mathcal{C}_{n}(\lambda_i(p'))) I(\lambda_i(p') = \lambda_j(p') \wedge \lambda_k(p'))$$

$$+ \lambda_i(p') I(\xi_{n+1} \in \mathcal{C}_{n}(\lambda_i(p'))) + s_i^{1+2}(s^1, s^2, s^3) I(\xi_{n+1} \in \mathcal{C}_{n}(\lambda_i(p'))) I(\lambda_i(p') = \lambda_j(p') \wedge \lambda_k(p'))$$

$$+ (\tau_i^{1+2}(s^1, s^2, s^3) I(\xi_{n+1} \in \mathcal{C}_{n}(\lambda_i(p'))) + \tau_i^{1+2}(s^1, s^2, s^3) I(\xi_{n+1} \in \mathcal{C}_{n}(\lambda_i(p'))) I(\lambda_i(p') = \lambda_j(p') \wedge \lambda_k(p'))$$

$$+ (\tau_i^{1+2}(s^1, s^2, s^3) I(\xi_{n+1} \in \mathcal{C}_{n}(\lambda_i(p'))) + \tau_i^{1+2}(s^1, s^2, s^3) I(\xi_{n+1} \in \mathcal{C}_{n}(\lambda_i(p'))) I(\lambda_i(p') = \lambda_j(p') \wedge \lambda_k(p'))$$

When the sets $\mathcal{C}_n$, $i = 1, 2, 3$, are defined by nonempty intervals we can formulate the effective stopping time for Player 1 as follows. Let $\bar{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_N)$ and $\bar{\delta} = (\delta_1, \delta_2, \ldots, \delta_N)$. For every triple $(s^1, s^2, s^3)$ such that $s^i \in \mathcal{M}^N_i$, we define

$$\tau_i(s^1, s^2, s^3) = \lambda_i(p') I(\bar{\gamma}(p') < \bar{\delta}(p') \wedge \lambda_i(p'))$$

$$+ (\lambda_i(p') I(\xi_{n+1} \in \mathcal{C}_{n}(\lambda_i(p'))) + s_i^{1+3}(s^1, s^2, s^3) I(\xi_{n+1} \in \mathcal{C}_{n}(\lambda_i(p'))) I(\lambda_i(p') = \lambda_j(p') \wedge \lambda_k(p'))$$

$$+ (\lambda_i(p') I(\xi_{n+1} \in \mathcal{C}_{n}(\lambda_i(p'))) + s_i^{1+3}(s^1, s^2, s^3) I(\xi_{n+1} \in \mathcal{C}_{n}(\lambda_i(p'))) I(\lambda_i(p') = \lambda_j(p') \wedge \lambda_k(p'))$$

$$+ (\tau_i^{1+3}(s^1, s^2, s^3) I(\xi_{n+1} \in \mathcal{C}_{n}(\lambda_i(p'))) + \tau_i^{1+3}(s^1, s^2, s^3) I(\xi_{n+1} \in \mathcal{C}_{n}(\lambda_i(p'))) I(\lambda_i(p') = \lambda_j(p') \wedge \lambda_k(p'))$$

$$+ (\tau_i^{1+3}(s^1, s^2, s^3) I(\xi_{n+1} \in \mathcal{C}_{n}(\lambda_i(p'))) + \tau_i^{1+3}(s^1, s^2, s^3) I(\xi_{n+1} \in \mathcal{C}_{n}(\lambda_i(p'))) I(\lambda_i(p') = \lambda_j(p') \wedge \lambda_k(p'))$$

where $\lambda_i$ is the time at which the $i$th player wishes to stop. This expression comes from considering which player (or players) wish to stop first and what happens when two or more players wish to stop at the same time. Similar expressions can be obtained for the stopping times of the other two players. The random variables $\tau_i(s^1, s^2, s^3) \in \mathcal{T}^N$ for every $s^i \in \mathcal{M}^N_i, i = 1, 2, 3$.

**Definition 2.** The Markov times $\tau_i(s^1, s^2, s^3)$ for $i = 1, 2, 3$ are the selection times of Player $i$ when they use strategies $s^i \in \mathcal{M}^N_i$, and the lottery is defined by $\bar{\gamma}$ and $\bar{\delta}$.

For each $(s^1, s^2, s^3) \in \mathcal{M}^N_i \times \mathcal{M}^N_j \times \mathcal{M}^N_k$ and given $\bar{\gamma}$ and $\bar{\delta}$, the payoff function for the $i$th player is defined as $f_i(s^1, s^2, s^3) = g_i(T_{\tau_i(s^1, s^2, s^3)})$. Let $R_i(x, s^1, s^2, s^3) = E_x f_i(s^1, s^2, s^3) = E_x g_i(T_{\tau_i(s^1, s^2, s^3)})$ be the expected gain of the $i$th player if the players use $(s^1, s^2, s^3)$. We have defined the game in normal form $(\mathcal{M}^N_1, \mathcal{M}^N_2, \mathcal{M}^N_3, \bar{R}_1, \bar{R}_2, \bar{R}_3)$. This random-priority game will be denoted $G_{rp}$.

**Definition 3.** A triple $(s^1, s^2, s^3)$ of strategies such that $s^i \in \mathcal{M}^N_i, i = 1, 2, 3$, is called a Nash equilibrium in $G_{rp}$ if for all $x \in E$

$$c_1(x) = \bar{R}_1(x, s^1, s^2, s^3) \geq \bar{R}_1(x, s, s^2, s^3) \text{ for every } s \in \mathcal{M}^N_1,$$

$$c_2(x) = \bar{R}_2(x, s^1, s^2, s^3) \geq \bar{R}_1(x, s^1, s, s^3) \text{ for every } s \in \mathcal{M}^N_2,$$

$$c_3(x) = \bar{R}_3(x, s^1, s^2, s^3) \geq \bar{R}_1(x, s^1, s^2, s) \text{ for every } s \in \mathcal{M}^N_3.$$

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The triple \((v_1(x), v_2(x), v_3(x))\) will be called the Nash value.

In [14], the two-person non-zero-sum game has been solved. After the first successful acceptance of a state in the three-person game, the players who have not accepted a state are playing such a two-person game. Taking into account the above definition of \(G_{WP}\), one can conclude that the Nash values of this game are the same as in the auxiliary game \(G_{WP}\) with the sets of strategies \(P^{N,1} \times P^{N,2} \times P^{N,3}\). Considering the times at which each player wishes to stop, the payoff of Player 1 can be expressed as

\[
\varphi_1(p^1, p^2, p^3) = g_1(X_{\lambda_1(p)} I_{\lambda_1(p') < \lambda_1(p) \land \lambda_2(p')}) + R_i^1 \varphi_1(X_{\lambda_1(p')}, p^1, p^2, p^3) I_{\lambda_1(p') < \lambda_1(p)} \land \lambda_2(p') + R_i^2 \varphi_1(X_{\lambda_1(p')}, p^1, p^2, p^3) I_{\lambda_1(p') = \lambda_1(p)} \land \lambda_2(p') + R_i^3 \varphi_1(X_{\lambda_1(p')}, p^1, p^2, p^3) I_{\lambda_1(p') = \lambda_1(p) \land \lambda_2(p')}
\]

When the lottery is defined by nonempty intervals and the stopping times of the players are as in (1), then the payoff of Player 1 can be expressed as

\[
\varphi_1(p^1, p^2, p^3) = g_1(X_{\lambda_1(p)} I_{\lambda_1(p') < \lambda_1(p) \land \lambda_2(p')}) + R_i^{1,2} \varphi_1(X_{\lambda_1(p')}, p^1, p^2, p^3) I_{\lambda_1(p') < \lambda_1(p) \land \lambda_2(p')}
\]

Expressions for the payoffs of the other two players can be obtained in a similar way. Let \(P^{N,i} = \{p = (p_i^1, \ldots, p_i^{n-1} = 0, p_i^n = 1), i = 1, 2, 3\}. \) We will use the following convention: if \( p^i \in P^{N,i} \), then \( (p_i^1, p^i) \) is the strategy belonging to \( P^{N,i} \) in which the n-th coordinate is changed to \( p_i^n \).

**Definition 4.** A triple \((p^1, p^2, p^3)\) in \(P^{N,1} \times P^{N,2} \times P^{N,3}\) is called an equilibrium point of \(G_{WP}\) at \(n\) if

\[
v_1(n, X_n) = E_{X_n} \varphi_1(p^1, p^2, p^3) \geq E_{X_n} \varphi_1(p^1, p^2, p^3) \text{ for every } p^1 \in P^{N,1}, \ P_{x-a.s.}
\]

\[
v_2(n, X_n) = E_{X_n} \varphi_2(p^1, p^2, p^3) \geq E_{X_n} \varphi_2(p^1, p^2, p^3) \text{ for every } p^2 \in P^{N,2}, \ P_{x-a.s.}
\]

\[
v_3(n, X_n) = E_{X_n} \varphi_3(p^1, p^2, p^3) \geq E_{X_n} \varphi_3(p^1, p^2, p^3) \text{ for every } p^3 \in P^{N,3}, \ P_{x-a.s.}
\]

A Nash equilibrium point at \(n = 0\) is a solution of \(G_{WP}\). The triple \((v_1(0, x), v_2(0, x), v_3(0, x))\) of values is a Nash value corresponding to \((p^1, p^2, p^3)\) in \(P^{N,1} \times P^{N,2} \times P^{N,3}\).

3. Solution of the Game

We introduce some preliminary notation. We use \(R_i^{1,2} = R_i^{1,2}(x, p^i, p^{i+2})\) and \(R_i^{1,3} = R_i^{1,3}(x, p^i, p^{i+2})\) for \(i = 1, 2, 3\). Assume that \(0 < \gamma_n < \delta_n < 1\). Let \(\alpha_n^1 = \gamma_n, \alpha_n^2 = \delta_n - \gamma_n, \alpha_n^3 = 1 - \delta_n, \alpha_n^4 = \alpha_n^1/(\alpha_n^1 + \alpha_n^2), \alpha_n^{2,3} = \alpha_n^2/(\alpha_n^1 + \alpha_n^2)\). Let

\[
g_i(x, x) = g_i(x) + \alpha_n^1 R_i^{1,2}(x, p^i, p^{i+2}) + \alpha_n^2 R_i^{1,3}(x, p^i, p^{i+2}).
\]
In the construction of the solution of the game, matrix games are solved. Assume that up to moment \( n \) no player has accepted any state. If, at moment \( n \), Player 1 chooses to stop and accept the state, then the gain matrix is as follows:

\[
\begin{bmatrix}
  (g_1, g_2, g_3) & (\alpha_n^{1,2}g_1 + (1 - \alpha_n^{1,2})R_1^{1,3}, \\
  \alpha_n^{1,2}R_2^{1,3} + (1 - \alpha_n^{1,2})g_2, \\
  \alpha_n^{1,2}R_3^{1,3} + (1 - \alpha_n^{1,2})g_3)
\end{bmatrix}
\]

(4)

and when Player 1 does not decide to stop and accept the state, then the gain matrix is as follows:

\[
\begin{bmatrix}
  (\alpha_n^{2,3}R_1^{2,3} + (1 - \alpha_n^{2,3})R_1^{1,2}, \\
  \alpha_n^{2,3}g_2 + (1 - \alpha_n^{2,3})R_2^{1,3}, \\
  \alpha_n^{2,3}g_2 + (1 - \alpha_n^{2,3})R_3^{1,3})
\end{bmatrix}
\]

(5)

**Theorem 1.** There exists a Nash equilibrium \((p^*, p^*, p^*)\) in the game \(G_{wp}\). The Nash value and equilibrium point can be calculated recursively.

**Proof.** At moment \( N \), the players play the three-person matrix game with gain matrices (4)–(5). This game always has an equilibrium in pure or randomized strategies on \( \omega: X_n = x \) for every \( x \in E \). We denote a Nash equilibrium in \( P_{n+1}^N \times P_{n+1}^N \times P_{n+1}^N \) by \((p_n^*, p_n^*, p_n^*)\) and the corresponding Nash value by \((v_1(N, x), v_2(N, x), v_3(N, x))\). Let us assume that an equilibrium \((p^*, p^*, p^*)\) in \( P_{n+1}^N \times P_{n+1}^N \times P_{n+1}^N \) has been constructed and \((v_1(n + 1, x), v_2(n + 1, x), v_3(n + 1, x))\) is the Nash value corresponding to this strategy on \( \omega: X_n = x \). We consider the three-person matrix game with gain matrices (4)–(5). On the set \( \omega: X_n = x \) there is at least one equilibrium point in pure or randomized strategies in this game. By the measurability of \( g_i(x) \), there exists \((p_n^*, p_n^*, p_n^*)\) such that \( p_n^* \in P_n^i, i = 1, 2, 3, \) and \((p_n^*, p_n^*, p_n^*)\) is a Nash equilibrium in the above game. We are now in a position to show that \( \bar{p}^* = ((p_n^*, p_n^*, p_n^*)^T, (p_n^*, p_n^*, p_n^*)^T) \) is an equilibrium of \( G_{wp} \) in \( P_{n+1}^N \times P_{n+1}^N \times P_{n+1}^N \). Define \( p_i^* = ((p_n^i, p_n^i, p_n^i)^T, (p_n^i, p_n^i, p_n^i)^T) \). Let \( p_i^* \) for \( i = 1, 2, 3 \) be defined analogously. By the properties of conditional expectation and the induction assumption, we have \( P_x \)-a.s.

\[
E_{X_n, \omega^1} \cdot p(p_i(p_i^*)^T) = p_{n+1}^{i+1}(g_n(X_{\lambda_i(n)})P\{\xi_n \in C_n^i \} + R_n^{i+1}(X_{\lambda_i(n)}, p_i^*)P\{\xi_n \in C_n^i \} \mid \xi_n \in C_n^i) + p_n^i (1 - p_n^{i+1})^* \cdot \lambda_i(n)^*(g_n(X_{\lambda_i(n)})P\{\xi_n \in C_n^i \} \mid \xi_n \in C_n^i) + R_n^{i+1}(X_{\lambda_i(n)}, p_i^*)P\{\xi_n \in C_n^i \} \mid \xi_n \in C_n^i)
\]

When the lottery is defined by nonempty intervals, we get for Player 1 (see (1))

\[
E_{X_n, \omega^1} \cdot p(p_i(p_i^*)^T) = p_{n+1}^{i+1}(g_n(X_{\lambda_i(n)})P\{\xi_n \in C_n^i \} \mid \xi_n \in C_n^i) + R_n^{i+1}(X_{\lambda_i(n)}, p_i^*)P\{\xi_n \in C_n^i \} \mid \xi_n \in C_n^i)
\]

When the lottery is defined by nonempty intervals, we get for Player 1 (see (1))

\[
E_{X_n, \omega^1} \cdot p(p_i(p_i^*)^T) = p_{n+1}^{i+1}(g_n(X_{\lambda_i(n)})P\{\xi_n \in C_n^i \} \mid \xi_n \in C_n^i) + R_n^{i+1}(X_{\lambda_i(n)}, p_i^*)P\{\xi_n \in C_n^i \} \mid \xi_n \in C_n^i)
\]

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when this candidate appears, we obtain the following integral expressions:

\[
W_{1,2}(t) = \int_{1}^{t} t[x \gamma x + (1 - \gamma)U(x)] dx,
\]
\[
W_{2,1}(t) = \int_{1}^{t} t[(1 - \gamma)x + \gamma U(x)] dx.
\]

Dividing by \( t \) and differentiating, we obtain the differential equations

\[
\frac{tW'_{1,2}(t) - W_{1,2}(t)}{t^2} = -\frac{\gamma}{t} + \frac{(1 - \gamma) \log t}{t},
\]
\[
\frac{tW'_{2,1}(t) - W_{2,1}(t)}{t^2} = -\frac{1 - \gamma}{t} + \frac{\gamma \log t}{t}.
\]

Together with the boundary conditions \( W_{1,2}(1) = W_{2,1}(1) = 0 \), this gives

\[
W_{1,2}(t) = -\gamma t \log t + \frac{(1 - \gamma)t \log^2 t}{2},
\]
\[
W_{2,1}(t) = -(1 - \gamma)t \log t + \frac{\gamma t \log^2 t}{2}.
\]

For \( t_{2,1} < t < e^{-1} \), it is stable for Player 2 to accept a candidate, while Player 1 rejects any applicant. Suppose Player 1 rejects a candidate who appears at time \( t \). By accepting such a candidate, Player 2 obtains an expected reward of \( t \); otherwise, he/she obtains an expected reward of \( W_{2,1}(t) \). Thus, Player 2 should accept a candidate if and only if \( t \geq W_{2,1}(t) \). In this interval, the reward functions are given by the differential equations

\[
W'_{1,2}(t) - \frac{W_{1,2}(t)}{t} = -e^{-1},
\]
\[
W'_{2,1}(t) - \frac{W_{2,1}(t)}{t} = -1.
\]

Using the boundary conditions at \( t = e^{-1} \), which result from the continuity of the reward functions, these equations give

\[
W_{1,2}(t) = e^{-1} - \frac{(1 - \gamma)t}{2},
\]
\[
W_{2,1}(t) = -\frac{\gamma t}{2} - t \log t.
\]

Using the stability condition given above, we obtain \( t_{2,1} = \exp[-\gamma/2 - 1] \). The reward functions are constant for \( t \leq t_{2,1} \).

Consider the subgames played between Player 3 and one of the other players. Player 3 never has priority, and this corresponds to setting \( \gamma = 1 \) in the equations for the subgame between Player 1 and Player 2. Thus, \( W_{2,1}(t) = U(t) \), \( t_{2,3} = e^{-1} \) together with

\[
W_{3,i}(t) = \begin{cases} 
\frac{t \log^2 t}{2}, & t \geq e^{-1}, \\
\frac{t}{2} - t \log t, & t < e^{-1},
\end{cases}
\]

and \( t_{3,i} = e^{-1.5} \) for \( i \in \{1, 2\} \).

Having considered all the subgames, we can now consider the 3-player game. In the case \( \delta = 1 \), the analysis is simplified, since the strategy of Player 3 does not have any effect on the reward of the other players. Thus \( V_1(t) = W_{1,2}(t), V_2(t) = W_{2,1}(t), t_1 = t_{1,2}, \) and \( t_2 = t_{2,1} \). Player 3 should be less choosy than the remaining players and so \( t_3 < t_2 \). Now Player 3's reward function in the 2-player subgames is the same, regardless of which other player remains.
for each $x \in E$. The same is valid for Player 2 and Player 3. This proves the theorem.

The solution of the game $G_{wp}$, based on the solution $(p^*, q^*)$ of the corresponding game $G_{wp}$, can be constructed.

The solutions of examples are based on the existence of the Nash value in the three-person game considered.

4. Examples

In this section, we present the solution of some games that are based on the secretary problem. The priority scheme is constant, i.e., $\gamma = \gamma_0 = \delta_0 = \delta$, $\forall n$. In each case, it is assumed that the number of objects (henceforth called applicants) presented to the players is very large ($N \to \infty$) and the object of each player is to obtain the best applicant. Thus, it is assumed that each player obtains a reward of 1 if he obtains the best applicant and 0 in all other cases. It is further assumed that each player has complete knowledge of the relative ranks of the applicants already seen. The $n$th applicant seen has relative rank $i$ ($i \in \{1, 2, \ldots, n\}$ if the $n$th applicant seen is the $i$th best applicant seen so far. In such cases, a player should only accept an applicant who has relative rank 1. These applicants will be referred to as candidates. If more than one player wishes to accept a candidate, then a lottery device decides which player has priority and assigns to each applicant, as previously described. If one player obtains a candidate, then he/she stops searching. The other players are informed of this and are allowed to continue searching.

Assume that $t = n/N$; thus $t \in [0, 1]$. We will find Nash equilibria for several examples. These solutions are found by recursion, working backwards from $t = 1$. These Nash equilibria will be described by a set of 10 times $t_0, t_{i,j}, t_k$ ($i, j \in \{1, 2, 3\}, i \neq j$), and $t_k$ ($k \in \{1, 2, 3\}$); $t_0$ and $U(t)$ describe the optimal policy and the optimal expected future reward, respectively, of an individual searcher after the other two searchers have employed an applicant. Such a player should accept the most valuable object seen so far if and only if $t \geq t_0$; $U(t)$ is the optimal expected reward of a lone searcher who is still searching at time $t$; $t_{i,j}$ and $W_{i,j}(t)$ describe the equilibrium policy and the expected future reward at equilibrium, respectively, of player $i$ in the subgame where Players $i$ and $j$ are still searching and the other player has employed an applicant. In this case, Player $i$ should accept a candidate if and only if $t \geq t_{i,j}$. Similarly, $t_i$ and $V_i(t)$ describe the equilibrium policy and expected future reward of the $i$th player when all three players are still searching. Note that these reward functions are equivalent to the operator $T_v(t)$ used in the game matrices presented in the previous section for the relevant function $v(t)$. In order to calculate $T_v(t)$, we need to consider the distribution of the time at which the next candidate appears after time $t$. Consider a finite number $N$ of applicants and suppose $n$ applicants have already been seen. It can be shown that the probability that the next candidate is the $k$th applicant ($k > n$) is $n/k(k-1)$. Note that the probability that no candidates will be seen in the future is $n/N$. Letting $N \to \infty$ and setting $t = n/N$, the probability density of the time $x$ ($x > t$) at which the next candidate is seen after time $t$ is $t/x^2$. With probability $t$ no more candidates are seen.

In general, each player should consider what happens in the subgames in which 1 or 2 players remain. So, in order to solve the 3-player game, we must first solve the 1- and 2-player subgames. From [4], the optimal policy of a lone searcher is to accept a candidate if and only if $t \geq e^{-1}$. Thus $t_0 = e^{-1}$; $U(t)$ is given by the following function:

$$U(t) = \begin{cases} -t \ln t, & t \geq e^{-1}, \\ e^{-1}, & t < e^{-1}. \end{cases}$$

Without loss of generality, we can assume that $t_1 > t_2 > t_3$. The first example given gives a general solution for the case $\delta = 1$. The second example is a specific example with $\delta < 1$, but consideration of the reward functions shows that the method of solution used is representative for this subclass of problems.

4.1. Example where one player never has priority. Let $0.5 < \gamma < 1$, $\delta = 1$. Thus, Player 1 is more likely to have priority than Player 2, and Player 3 never has priority if one of the other players wishes to accept a candidate. Firstly, we consider the game in which Players 1 and 2 are left searching. For large $t$ it will be stable for both players to accept a candidate. Suppose Player 2 wishes to accept a candidate, appearing at time $t$. By accepting such a candidate, Player 1 obtains an expected reward of $t + (1 - \gamma)U(t)$ and by rejecting such a candidate he/she obtains an expected reward of $U(t)$. It can thus be seen that Player 1 should accept a candidate if and only if $t \geq U(t)$, i.e., $t \geq e^{-1}$. Hence $t_{1,2} = e^{-1}$. By considering the distribution of the time of finding the next candidate after time $t$ and what happens
Thus, considering the distribution of the time at which the next candidate is found and what happens on finding this candidate, for \( t \geq t_2 \) we obtain the integral expression

\[
V_3(t) = \frac{1}{t} \int_1^t \frac{tW_{3,1}(x) \, dx}{x^2}.
\]

For \( t \geq e^{-1} \), this leads to

\[
V_3'(t) - \frac{V_3(t)}{t} = -\frac{\log^2 t}{2}.
\]

Together with the boundary condition \( V_3(1) = 0 \), this gives

\[
V_3(t) = -\frac{t \log^3 t}{6}.
\]

For \( t_2 < t < e^{-1} \), this integral expression leads to

\[
V_3'(t) - \frac{V_3(t)}{t} = \frac{2 \log t + 1}{2}.
\]

Together with the boundary condition at \( t = e^{-1} \), this gives

\[
V_3(t) = \frac{t[1 + 3 \log t + 3 \log^2 t]}{6}.
\]

Proceeding in a similar way for \( t_3 < t < t_2 \), we obtain

\[
V_3'(t) - \frac{V_3(t)}{t} = -1.
\]

Together with the boundary condition at \( t = t_2 \), this gives

\[
V_3(t) = kt - t \log t,
\]

where \( k = -5/6 - \gamma/(3 - \gamma)/8 \). If a candidate appears at time \( t < t_2 \), Player 3 obtains an expected reward of \( t \) by accepting such a candidate and \( V_3(t) \) by rejecting such a candidate. Thus, Player 3 should accept a candidate as long as \( t \geq V_3(t) \). This gives \( t_3 = \exp(k - 1) \). For \( t < t_3 \), \( V_3(t) = t_3 \).

4.2. Example with fixed priorities. Suppose \( \gamma = 1/2 \) and \( \delta = 4/5 \). In the subgame played between Players 1 and 2, Player 4 has priority with probability \( 5/8 \). From the analysis carried out in Example 1, we get \( t_{1,2} = e^{-1}, t_{2,1} = \exp(-21/16) \),

\[
W_{1,2}(t) = \begin{cases} 
\frac{3t \log^2 t - 10t \log t}{16}, & t \geq t_{1,2}, \\
e^{-1} - \frac{3t}{16}, & t_{2,1} \leq t < t_{1,2}, \\
e^{-1} - \frac{3t_{2,1}}{16}, & t < t_{2,1}, 
\end{cases}
\]

\[
W_{2,1}(t) = \begin{cases} 
\frac{5t \log^2 t - 6t \log t}{16}, & t \geq t_{1,2}, \\
\frac{5t}{16} - t \log t, & t_{2,1} \leq t < t_{1,2}, \\
t_{2,1}, & t < t_{2,1}.
\end{cases}
\]

In the subgame played between Player 1 and Player 3, Player 1 has priority with probability \( 5/7 \). This gives \( t_{1,3} = e^{-1}, t_{3,1} = \exp(-19/14) \),

\[
W_{1,3}(t) = \begin{cases} 
\frac{t \log^2 t - 5t \log t}{7}, & t \geq t_{1,3}, \\
e^{-1} - \frac{t}{7}, & t_{3,1} \leq t < t_{1,3}, \\
e^{-1} - \frac{t_{3,1}}{7}, & t < t_{3,1}, 
\end{cases}
\]

\[
W_{3,1}(t) = \begin{cases} 
\frac{5t \log^2 t - 4t \log t}{14}, & t \geq t_{1,3}, \\
\frac{5t}{14} - t \log t, & t_{3,1} \leq t < t_{1,3}, \\
t_{3,1}, & t < t_{3,1}.
\end{cases}
\]
In the subgame played between Player 2 and Player 3, Player 2 has priority with probability 3/5. This leads to $t_{2,3} = e^{-1}$, $t_{3,2} = \exp(-13/10)$,

$$W_{2,3}(t) = \begin{cases} \frac{t \log^2 t - 3t \log t}{5}, & t \geq t_{2,3}, \\ e^{-1} - \frac{t}{5}, & t_{3,2} \leq t < t_{2,3}, \\ e^{-1} - \frac{t_{3,2}}{5}, & t < t_{3,2} \end{cases}$$

$$W_{3,2}(t) = \begin{cases} \frac{3t \log^2 t - 4t \log t}{10}, & t \geq t_{2,3}, \\ -\frac{3t}{10} - t \log t, & t_{3,2} \leq t < t_{2,3}, \\ t_{3,2}, & t < t_{3,2} \end{cases}$$

Now we can consider the 3-person game. Suppose Players 2 and 3 wish to accept a candidate at time $t$. By accepting such a candidate, Player 1 obtains an expected reward of $[5t + 3W_{1,3}(t) + 2W_{1,2}(t)]/10$; otherwise, he/she obtains a reward of $[3W_{1,3}(t) + 2W_{1,2}(t)]/5$. Thus, Player 1 should accept such a candidate if and only if $t \geq [3W_{1,3}(t) + 2W_{1,2}(t)]/5$. Since $W_{i,j}(t) < U(t)$ for $i, j \in \{1, 2, 3\}$, it follows that the above inequality is satisfied for $t \geq e^{-1}$. Also, for $t < 1$ we have $W_{1,i}(t) > W_{i,1}(t)$ for $i \in \{2, 3\}$ and $W_{1,i}(t) > W_{3,2}(t)$. Thus, it can be seen that $t_1 > \max\{t_{2,1}, t_{3,1}, t_{3,2}\}$. This result is true for all problems of this type. In the problem, the considered $t_1$ fulfills the following equation:

$$t_1 = \frac{3}{5} \left( e^{-1} - \frac{t_1}{7} \right) + \frac{2}{5} \left( e^{-1} - \frac{3t_1}{16} \right).$$

This gives $t_1 = 56e^{-1}/65$. For $t > t_1$, we have

$$V'_1(t) - \frac{V_1(t)}{t} = -\frac{1}{2} \frac{3W_{1,2}(t) + 2W_{1,2}(t)}{10t},$$

$$V'_2(t) - \frac{V_2(t)}{t} = -\frac{3}{10} \frac{5W_{2,3}(t) + 2W_{2,1}(t)}{10t},$$

$$V'_3(t) - \frac{V_3(t)}{t} = -\frac{1}{5} \frac{5W_{3,2}(t) + 3W_{3,1}(t)}{10t}.$$

Together with the boundary conditions $V_i(1) = 0$, $i \in \{1, 2, 3\}$, for $t \geq e^{-1}$, these equations lead to

$$V_1(t) = \frac{t[19 \log^2 t - 56 \log t - 3 \log^3 t]}{112},$$

$$V_2(t) = \frac{t[45 \log^2 t - 72 \log t - 13 \log^3 t]}{240},$$

$$V_3(t) = \frac{t[5 \log^2 t - 7 \log t - 3 \log^3 t]}{35},$$

and for $t_1 < t \leq e^{-1}$,

$$V_1(t) = e^{-1} - \frac{t[25 + 47 \log t]}{112},$$

$$V_2(t) = e^{-1} - \frac{t[47 + 33 \log t - 21 \log^2 t]}{240},$$

$$V_3(t) = \frac{t[3 + 2 \log t + 14 \log^2 t]}{35}.$$

Suppose a candidate appears at time $t$, where $t_2 < t < t_1$. By accepting such a candidate, Player 2 obtains an expected reward of $[3t + 2W_{2,1}(t)]/5$ and by rejecting such a candidate obtains a reward of $W_{2,1}(t)$. Thus, Player 2 should accept candidates if and only if $t \geq W_{2,1}(t)$. Thus, $t_2 = t_{2,1} = \exp(-21/16)$. Note there is no possibility of the
3-player game switching to the subgame played between Players 2 and 3 when \( t < t_1 \). Since we also have \( t_3,t < t_2 \) (this is true for a general problem), the following differential equations hold on the interval \((t_2,t_1)\):

\[
\begin{align*}
V'_1(t) - \frac{V_1(t)}{t} &= -\frac{3W_{1,2}(t) + 2W_{1,3}(t)}{5t}, \\
V'_2(t) - \frac{V_2(t)}{t} &= -\frac{3}{5} - \frac{2W_{2,1}(t)}{5t}, \\
V'_3(t) - \frac{V_3(t)}{t} &= -\frac{2}{5} - \frac{3W_{3,1}(t)}{5t}.
\end{align*}
\]

Together with the boundary conditions at \( t_1 \), these equations lead to

\[
\begin{align*}
V_1(t) &= e^{-1} + \frac{9t \log t}{56} - c_1 t, \\
V_2(t) &= c_2 t - \frac{19t \log t}{40} + \frac{t \log^2 t}{5}, \\
V_3(t) &= c_3 t - \frac{13t \log t}{70} + \frac{3t \log^2 t}{10},
\end{align*}
\]

where

\[
\begin{align*}
c_1 &= \frac{90 + 65[\log(56/65) - 1]}{112}, \\
c_2 &= \frac{646 + 567[\log(56/65) - 1] - 168[\log(56/65) - 1]^2}{1680}, \\
c_3 &= \frac{6 + 17[\log(56/65) - 1] + 7[\log(56/65) - 1]^2}{70}.
\end{align*}
\]

Suppose a candidate appears in the interval \((t_3,t_2)\). By accepting such a candidate, Player 3 obtains an expected reward of \( t \) and by rejecting such a candidate obtains an expected reward of \( V_3(t) \). Thus, Player 3 should accept a candidate as long as \( t \geq V_3(t) \). On this interval:

\[
\begin{align*}
V'_1(t) - \frac{V_1(t)}{t} &= -\frac{W_{1,2}(t)}{t}, \\
V'_2(t) - \frac{V_2(t)}{t} &= -\frac{W_{2,1}(t)}{t}, \\
V'_3(t) - \frac{V_3(t)}{t} &= -1.
\end{align*}
\]

Together with the boundary conditions at \( t_2 \), these equations lead to

\[
\begin{align*}
V_1(t) &= e^{-1} - \frac{3t_2}{16} - \frac{t[741 + 520(\log(56/65) - 1)]}{896}, \\
V_2(t) &= c_4 t + t_2, \\
V_3(t) &= c_5 t - t \log t,
\end{align*}
\]

where

\[
\begin{align*}
c_4 &= \frac{1791}{5360} + \frac{[\log(56/65) - 1][27 - 8(\log(56/65) - 1)]}{80}, \\
c_5 &= \frac{[\log(56/65) - 1][17 + 7(\log(56/65) - 1)]}{1671} - \frac{1}{3584}.
\end{align*}
\]

This gives \( t_3 = \exp(c_5 - 1) \). For \( t < t_3 \), the reward functions are constant. For numerical comparison, the thresholds and the values of the game to each player are given below to four decimal places: \( t_1 = 0.3169, V_1(0) = 0.2855, t_2 = 0.2691, V_2(0) = 0.2322, \) and \( t_3 = V_3(0) = 0.1992 \).

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