MODIFIED STRATEGIES IN TWO PERSON
FULL-INFORMATION BEST CHOICE PROBLEM
WITH IMPERFECT OBSERVATION

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ABSTRACT. A zero-sum game version of the full-information best choice problem is considered. Two players observe sequentially a stream of iid random variables from a known continuous distribution appearing, according to some renewal process, with the object of choosing the largest one. The horizon of observation is a positive random variable independent of the observations. The observations of the random variables are imperfect and a player is informed only whether a random variable is greater than or less than some level he has previously specified. If one Player accepts an observation, the other Player can change his level and continue the game alone. A similar game with discrete time and a random number of observations is considered as a dual problem. The normal form of the game is derived. The value of the game and the form of the equilibrium strategy are obtained for the model with a Poisson stream of observations and exponentially distributed time horizon. In the discrete-time case a game with geometric number of observations is completely solved.

1. Introduction. This paper deals with the following zero-sum game version of the continuous-time full-information best choice problem. Two players observe sequentially a stream of iid random variables from a known continuous distribution appearing according to some renewal process with the object of choosing the largest. A choice must be made before the moment $T$, which is a positive random variable independent of the observations. The random variables cannot be perfectly observed. Each time a random variable is sampled, the sampler is informed only whether it is greater than or less than some level he has specified. Each Player can choose at most one observation. After each sampling, players make a decision to accept or reject the observation. If both want to accept the same observation, priority is given to a specified player, say Player 1. At the moment when one Player accepts an observation, the other one can change his level and continue the game alone.

The class of adequate strategies and the relevant gain function for the problem is constructed. The natural case of the model involving a Poisson renewal process

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with parameter \( \lambda \) and exponentially distributed \( T \) with parameter \( \mu \) is examined in detail. The value of the game and the optimal strategy depends on \( \lambda \) and \( \mu \) through \( p = \mu / (\mu + \lambda) \) only.

2. The priority game with imperfect observation. Let \( \xi_1, \xi_2, \xi_3, \ldots \) be a sequence of iid random variables from a common and known continuous distribution \( F \) defined on a probability space \((\Omega, \mathcal{F}, P)\). \( \xi \)'s appear according to some renewal process and let \( \rho_n \) stand for the length of the time interval between \( \xi_{n-1} \) and \( \xi_n \) (for convenience it is assumed \( \xi_0 = 0 \) by definition), i.e. \( \rho_1, \rho_2, \rho_3, \ldots \) are iid positive random variables with a continuous distribution \( G \). A positive random variable \( T \) with a distribution \( H \) represents the moment, when the observation is terminated. The \( \xi \)'s, \( \rho \)'s and \( T \) are independent.

The sequence of random variables is sequentially sampled one by one by two decision makers (players). However, the observations are imperfect and the exact realized values are not known. Players specify only their level of sensitivity (impressionability) and they are able to know whether the observed random variable is greater than or less than their prescribed levels, chosen individually. After \( \xi_n \) is observed, Player 1 has to accept or reject the observation. If he rejects it, then Player 2 has to decide whether to reject the observed realization. When one player accepts an observation, then the other investigates the sequence of future realizations, having the opportunity of accepting one of them. After the first acceptance, the player, who has not yet accepted an observation, can change his level of sensitivity (i.e. can modify his strategy). Neither recall nor uncertainty of selection is allowed. One can say that Player 1 has priority in accepting a realization. The aim of the players is to choose the best observation (the maximal one).

Reviews of related one-person problems and results for discrete time one can find in the papers by Sakaguchi [8] and Porosiński [5]. Let us mention here the papers concerning variation of the best choice problem with Poisson arrivals, considered by Cowan & Zabczyk [1], Enns & Ferenstein [4] and Dixon [2].

In this paper we model the problem as two person zero-sum game. Similar models for the no-information case have been considered, e.g. in Szajowski [9] and Enns & Ferenstein [3]. The continuous time zero-sum game related to the full-information best choice problem with imperfect observation, has been solved by the authors in Porosiński & Szajowski [6]. Since changes in the specified levels determining the strategies of the players appear during the games presented here, the structure of the strategy sets and the form of the gain functions are different to those considered by Porosiński & Szajowski [6].

Let

\[
S_n = \rho_1 + \ldots + \rho_n, \quad n = 1, 2, \ldots, \quad S_0 = 0,
\]

\[
N(t) = \max\{n \geq 0 : S_n \leq t\}, \quad t \geq 0.
\]

Since \( S_n \) is the waiting time until the \( n \)-th observation and \( N(t) \) is the total number of \( \xi \)'s that appeared up to time \( t \). At the moment when \( \xi_n \) is observed, all previous values of \( \xi \)'s and \( \rho \)'s are known and moreover it is known whether the moment \( T \) follows or not i.e. the \( \sigma \)-field of information is

\[
\mathcal{F}_n = \sigma\{\xi_1, \ldots, \xi_n, \rho_1, \ldots, \rho_n, \chi_{\{T > S_n\}}, \ldots, \chi_{\{T > S_{n-1}\}}\}, \quad n = 1, 2, \ldots.
\]
where $\chi_A$ stands for the indicator function of the event $A$.

Let $S$ be the set of stopping times with respect to $\{F_n\}_{n=0}^\infty$. Since the observations are imperfect, $S^0 = \{\tau \in S : \tau = \inf\{n \leq N(T) : \xi_n \geq x\}, x \in \mathbb{R}\}$ is the class of adequate strategies for the one person decision problem. This set of strategies is not appropriate for the two person game, since the behavior of the player, who remains in the game after the first acceptance, depends on the information available when the first acceptance was made. The following modification of the strategy set solves this problem (see Szajowski [9]).

Let $S^i_k = \{\tau \in S^0 : \tau \geq k\} (S^0 = S^0_0)$. Define the strategy set $\mathcal{T}_i = \{(\sigma^i_0, \{\sigma^i_n\}) : \sigma^i_0 \in S^0, \sigma^i_n \in S^0_{n+1} \text{ for } n \geq 1\}, i = 1, 2$, for Player 1 and 2, respectively. When Player 1 chooses $x \in \mathcal{T}_1$ and Player 2 chooses $y \in \mathcal{T}_2$ then the effective stopping times $\tau_1 = \sigma^1_0 \chi_{\{\sigma^1_0 \leq \sigma^1_1\}} + \sigma^1_0 \chi_{\{\sigma^1_0 > \sigma^1_1\}}$ and $\tau_2 = \sigma^2_0 \chi_{\{\sigma^2_0 \leq \sigma^2_1\}} + \sigma^2_0 \chi_{\{\sigma^2_0 > \sigma^2_1\}}$ are defined for both of them.

Since the aim is to choose the best observation, let

$$f(x, y) = P(\xi_{\tau_1} = \max\{\xi_1, \ldots, \xi_{N(T)}\}) - P(\xi_{\tau_2} = \max\{\xi_1, \ldots, \xi_{N(T)}\})$$

be the payoff function.

From now on, without loss of generality, we assume that the observed random variables come from the standard uniform distribution (to study the general case, put $F(\xi_n)$ instead of $\xi_n$).

**Definition 1.** A pair $(x^*, y^*) \in \mathcal{T}_1 \times \mathcal{T}_2$ is an equilibrium point in the considered game, if for every $x \in \mathcal{T}_1$ and $y \in \mathcal{T}_2$ we have $f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y)$.

In fact, the players choose their levels of sensitivity at the beginning of the game and they keep these strategies up to the first acceptance. If Player 1 accepts an observation first, at that moment Player 2 may change his strategy by choosing a new level. He knows only the strategy of Player 1 (the exact value of chosen observation is unknown) and he uses this information when he decides which level to choose. Similarly, if Player 2 accepts an observation first, then Player 1 changes his strategy based on his knowledge about decision process so far.

Let Player 1 and Player 2 choose the levels $x \in [0, 1]$ and $y \in [0, 1]$, respectively. If $x < y$ and Player 1 accepts $\xi_s$, then Player 2 changes level to $y_s = y_s(x, 1)$. Player 1 gets +1 when $\{N(T) \geq s, \xi_1 < x, \ldots, \xi_{s-1} < x, \xi_s \geq x, \xi_{s+1} < \xi_s, \ldots, \xi_{N(T)} < \xi_s\}$, -1 when there is $t > s$ such that $\{N(T) \geq t, \xi_1 < x, \ldots, \xi_{t-1} < x, \xi_t \geq x, \xi_{t+1} < y_s, \ldots, \xi_s < y_s, \xi_{s+1} < \xi_s, \ldots, \xi_{N(T)} < \xi_t\}$ where $x \lor y = \max\{x, y\}$ and 0 otherwise. Let $x \geq y$. Suppose Player 1 accepts $\xi_s$ before Player 2 has accepted an observation. In this case Player 2 chooses a new level $y_s = y_s(x, 1)$. Player 1 gets +1 when $\{N(T) \geq s, \xi_1 < y, \ldots, \xi_{s-1} < y, \xi_s \geq x, \xi_{s+1} < \xi_s, \ldots, \xi_{N(T)} < \xi_s\}$ and -1, if there is $t > s$ such that $\{N(T) \geq t, \xi_1 < y, \ldots, \xi_{t-1} < y, \xi_t \geq x, \xi_{t+1} < y_s, \ldots, \xi_s \geq y_s, \xi_{s+1} < \xi_s, \ldots, \xi_{N(T)} < \xi_t\}$, otherwise Player 1 gets 0. Now suppose Player 2 accepts $\xi_s$ before Player 1 has accepted an observation. In this case Player 1 chooses new level $x_s = x_s(y, x)$ (he knows that $\xi_s \in \{y, x\}$). His payoff is +1, if there exists a $t > s$ such that $\{N(T) \geq t, \xi_1 < y, \ldots, \xi_{s-1} < y, \xi_t \geq x, \xi_{t+1} < x_s, \xi_s \geq \xi_t, \xi_{s+1} < \xi_s, \ldots, \xi_{N(T)} < \xi_t\}$, and
−1 when \( \{N(T) \geq s, \xi_1 < y, \ldots, \xi_{s-1} < y, y \leq \xi_s < x, \xi_{s+1} < \xi_s, \ldots, \xi_{N(T)} < \xi_s \} \). Otherwise his payoff is 0.

Since the conditional distribution of \( \xi \), given \( \xi \in [a,b] \), is uniform on \([a,b]\), the level \( x_s(a,b) [y_s(a,b)] \) is the optimal strategy in one person best choice problem for Player 1 [Player 2] given \( N(T) \geq s \), when the opponent has chosen an observation \( \xi_s \in [a,b] \). In this way, the strategies \( x = (x, \{x_n\}) \in T_x \), \( y = (y, \{y_n\}) \in T_y \) are constructed and the problem is reduced to the zero-sum game on the unit square (\( S^0 \) is equivalent to the interval \([0,1]\)), with the payoff function being the expected value of Player 1’s gain.

3. Equivalent game on unit square. Let \( f_n(x, y) \) denote the payoff to Player 1, when the levels \( (x, y) \) are chosen and \( N(T) = n \). Then for \( x < y \) we have

\[
\begin{align*}
f_n(x, y) &= \sum_{s=1}^{n} x^{s-1} \int_{x}^{1} \xi_s^{n-s} d\xi_s \\
&\quad - \sum_{t=2}^{n} \sum_{s=1}^{t-1} x^{s-1} \int_{x}^{y} y_s^{t-s-1} d\xi_s \int_{y, \xi_s}^{1} \xi_t^{n-t} d\xi_t
\end{align*}
\]

and for \( x \geq y \)

\[
\begin{align*}
f_n(x, y) &= \sum_{s=1}^{n} y^{s-1} \int_{x}^{1} \xi_s^{n-s} d\xi_s - \sum_{s=1}^{n} y^{s-1} \int_{y}^{x} \xi_s^{n-s} d\xi_s \\
&\quad + \sum_{s=1}^{n} \sum_{t=s+1}^{n} y^{s-1} \int_{y}^{x} x_s^{t-s-1} d\xi_s \int_{x, \xi_s}^{1} \xi_t^{n-t} d\xi_t \\
&\quad - \sum_{s=1}^{n} \sum_{t=s+1}^{n} y^{s-1} \int_{x}^{1} y_s^{t-s-1} d\xi_s \int_{y, \xi_s}^{1} \xi_t^{n-t} d\xi_t
\end{align*}
\]

where \( x_s = x_s(y, x) [y_s = y_s(x, 1)] \) is an optimal level for Player 1 [Player 2] starting from moment \( s \), if he knows that \( N \geq s \) and the opponent has chosen some observation \( \xi_s \in [y, x] \) in \([x, 1]\) according to the uniform distribution.

The payoff function is then \( f(x, y) = \sum_{n=1}^{\infty} f_n(x, y) P(N(T) = n) \), where \( f_n(x, y) \) is given by (3.1).

Based on the distributions of \( T \) and the process \( N(t) \), the distribution of the total number of observations can be found

\[
P(N(T) = n) = \int_{0}^{\infty} P(S_n \leq t, S_{n+1} > t) dH(t)
\]

\[
= \int_{0}^{\infty} dH(t) \int_{0}^{t} P(S_{n+1} > t | S_n = s) dG^*(s)
\]

\[
= \int_{0}^{\infty} dH(t) \int_{0}^{t} P(\rho_{n+1} > t - s) dG^*(s),
\]

where \( G^* \) stands for the distribution of \( S_n \).

Due to the form of the payoff function, it is very difficult to obtain the optimal levels \( (x, y) \) explicitly, even if the distributions of \( G \) and \( H \) are fixed. Nevertheless, in a natural case considered below, the solution has a very simple form.
4. A Poisson stream of observations. Let $G$ be exponential with parameter $\lambda$. Thus $(N(t))_{t \in [0, +\infty)}$ is the Poisson process with parameter $\lambda$. Moreover, let $T$ have an exponential distribution with parameter $\mu$. In this case, we have for $s \geq 0$

\[ dG^x(s) = \frac{\lambda^n}{(n-1)!} s^{n-1} e^{-\lambda s} ds \]

and the probability that exactly $N$ observations appear up to time $T$, given by (3.2), can be calculated as

\[ P(N(T) = n) = \int_0^\infty \left( \int_0^t e^{-\lambda(t-s)} \frac{\lambda^n}{(n-1)!} s^{n-1} e^{-\lambda s} ds \right) \mu e^{-\mu t} dt \]

\[ = \int_0^\infty \frac{\lambda^n}{n!} \mu t^n e^{-(\lambda + \mu)t} dt = \frac{\mu \lambda^n}{(\lambda + \mu)^{n+1}}. \]

Since Players can choose observations at the moments they appear and the payoff function depends just on the number of observations, this problem is similar to the two-person problem of choosing the best observation with discrete time and random number of observations $N$ (i.e. $N = N(T)$).

So, the continuous-time game, considered in this section, is equivalent to the discrete-time problem, with the number of observations being a geometric random variable with parameter $p = \mu/(\lambda + \mu)$ (cf Porosiński & Szajowski [6] and Porosiński & Szajowski [7]).

5. The geometric case. Let $N$ have a geometric distribution with parameter $p$, i.e. $P(N = n) = pq^n$, $n = 0, 1, 2, \ldots$

In this case, the levels $x_1, y_s$ do not depend on the moment $s$, because the distribution of $N$ given $N \geq s$ is geometric with parameter $p$ for each $s$.

This enables us to change the order of summation in (3.1). After summing and integrating we can transform the payoff function into the form

\[ f(x, y) = \left\{ \begin{array}{ll}
\int_0^p q_x & g(x, 1, z_2) & \text{if } x < y, \\
\int_0^p q_y & (g(x, 1, z_2) - g(y, x, z_1)) & \text{if } x \geq y,
\end{array} \right. \]

where $z_1 = x_s$, $z_2 = y_s$ and

\[ g(a, b, z) = \int_a^b \left( \frac{p}{1 - q_x} + \frac{p}{1 - q_z} \ln \frac{p}{1 - q(z \vee \xi_s)} \right) d\xi_s. \]

The payoff function $f(x, y)$ can be written, after simplifications, as a function $\bar{f}(s, t)$ of new coordinate variables $s = p/(1 - qx)$, $t = p/(1 - qy)$

\[ \bar{f}(s, t) = \left\{ \begin{array}{ll}
s \bar{g}(s, 1, u_2) & \text{if } s < t, \\
t (\bar{g}(s, 1, u_2) - \bar{g}(t, s, u_1)) & \text{if } t \leq s,
\end{array} \right. \]

(5.1)

where $u_i = p/(1 - qz_i)$ and

\[ \bar{g}(A, B, u) = \left\{ \begin{array}{ll}
\ln B - \ln A - \frac{u}{B} (1 + \ln B) + \frac{u}{A} (1 + \ln A) & \text{if } p \leq u < A, \\
\ln B - \ln A - \frac{u}{B} (1 + \ln B) + \frac{u}{A} \ln u + 1 & \text{if } A \leq u < B, \\
\ln B - \ln A + u(\frac{1}{A} - \frac{1}{B}) \ln u & \text{if } B \leq u \leq 1.
\end{array} \right. \]
This transformation \([0, 1] \times [0, 1] \) onto \([p, 1] \times [p, 1] \) preserves monotonicity. Since Player 1 [Player 2] wants to choose a level \(u_1 [u_2] \) to maximize [minimize] \(f(s, t)\), we have to find the \(u^* = u^*(A, B)\) for which \(\min_{u \in [p, 1]} \tilde{g}(A, B, u)\) is achieved.

It is easy to check that such an optimal \(u^*\) has the following form (cf Porosinski & Szajowski [7]):

\[
(5.2) \quad u^* = \begin{cases} 
e^{-1} & \text{if } B < e^{-1}, \\ e^{-1} + A(1 + \ln B)/B & \text{if } B \geq e^{-1}. \end{cases}
\]

In the case \(B \geq e^{-1}\), it is easy to see that \(u^* \in [A \vee e^{-1}, B]\) and \(u^* = e^{A-1}\) for \(B = 1\). Then

\[
\tilde{g}(A, B, u^*) = \begin{cases} 
\ln B - \ln A - \left(\frac{1}{A} - \frac{1}{B}\right)e^{-1} & \text{if } A \leq B < e^{-1}, \\
\ln B - \ln A - \frac{1}{B}(e^{-1} - A(1 + \ln B)) + 1 & \text{if } B \geq e^{-1},
\end{cases}
\]

and \(\tilde{g}(A, 1, u^*) = -\ln A - e^{A-1}/A + 1\).

It is very interesting and important that only domain of the gain function \(\tilde{f}(s, t)\) is dependent on \(p\) (the form given by (5.1) is independent of \(p\)).

From the above considerations, the game is transformed to the zero-sum game on \([p, 1] \times [p, 1]\), with the gain function obtained from (5.1) by putting \(u_1 = u^*(t, s), u_2 = u^*(s, 1)\)

\[
\tilde{f}(s, t) = \begin{cases} 
s - s \ln s - e^{s-1} & \text{if } s < t, \\
t(\ln t - 2 \ln s - \frac{1}{s}e^{s-1}) + t + (1 - \frac{t}{s})e^{-1} & \text{if } t \leq s < e^{-1}, \\
t(\ln t - 2 \ln s - \frac{1}{s}e^{s-1}) + e^{-1 + (1 + \ln s)/s} & \text{if } s \geq t \vee e^{-1}.
\end{cases}
\]

Let us consider the function \(\tilde{f}(s, t)\) on the unit square.

**Lemma 1.** The function \(\tilde{f}(s, t)\) considered on \([0, 1] \times [0, 1]\) has the following properties:

1. is continuous in both variables;
2. for fixed \(s\), as a function of \(t\)
   a. is decreasing on \([0, s]\) if \(s \leq \bar{s} \approx 0.2320\) or has the unique minimum on \([0, s]\) if \(s > \bar{s}\) (\(\bar{s}\) is a solution of the equation \(\tilde{f}_t(s, \bar{s}) = 0\), i.e. \(2\bar{s} - e^{\bar{s}-1} = 0\));
   b. is constant on \([s, 1]\);
3. for fixed \(t\), as a function of \(s\)
   a. is increasing on \([0, t]\) and has the unique maximum on \([t, 1]\) at \(\bar{t}\) (independent of \(t\)) if \(t \leq \bar{t} \approx 0.3533\) (\(\bar{t}\) is a solution of the equation \(\tilde{f}_s(t, \bar{t}) = 0\) for \(t \leq s \leq e^{-1}\), i.e. \((1 - \bar{t})e^{1-1} + e^{-1} - 2\bar{t} = 0\));
   b. is increasing on \([0, t]\) and decreasing on \([t, 1]\) if \(t < \bar{t} \approx 0.5339\) (\(\bar{t}\) is a solution of the equation \(\tilde{f}_s(t, \bar{t}) = 0\) for \(s \leq t\) i.e. \(\ln \bar{t} + e^{1-1} = 0\));
   c. if \(t > \bar{t}\), \(\tilde{f}(s, t)\) has a maximum on \([0, t]\) at \(\bar{t}\) (independent of \(t\)) and is decreasing on \([t, 1]\).
The game with the gain function $\tilde{f}(s,t)$, considered on unit square, has the following solution. Player 1 has an optimal pure strategy $s^* \approx 0.3533$ and Player 2 has an optimal pure strategy $t^* = (s^*)^2 \exp(e^{s^*-1}) \approx 0.2107$.

The point $(s^*, t^*)$ is calculated as the unique solution of the equations $\tilde{f}_s(s,t) = 0$, $\tilde{f}_t(s,t) = 0$ in a triangle $t < s < e^{-1}$, i.e.

$$
(1 - s)e^{s-1} + e^{-1} - 2s = 0, \\
2s - 2s \ln s + s \ln t - e^{s-1} - e^{-1} = 0.
$$

At $(s^*, t^*)$ there is a saddle point of $\tilde{f}(s,t)$ on the unit square. The value of the game is $\tilde{f}(s^*, t^*) \approx 0.3677$. The pair of optimal strategies $(s^*, t^*)$ fulfills the minimax conditions

$$
\min_t \tilde{f}(s^*, t) = \max_s \tilde{f}_s(s, t), \quad \max_t \tilde{f}(s^*, t^*) = \min_s \tilde{f}_s(s, t).
$$

This solution on $[0,1] \times [0,1]$ is also valid on $[p,1] \times [p,1]$ for $p \leq t^* \approx 0.2107$. For $p > t^*$, the players have to modify their strategies according to the properties of $\tilde{f}(s,t)$ on $[p,1] \times [p,1]$. For $p \in (t^*, s^*)$ an equilibrium point is $(s^*, p)$ and for $p > s^*$ the conditions (5.3) are valid for $(p, p)$ (see Fig. 1).

For the main problem, based on the auxiliary game, we can formulate

**Proposition 1.** For geometric $N$ with parameter $p$ there exists a solution of the game in pure strategies. It has the following form depending on $p$:

1. If $p \leq t^* \approx 0.2107$, then the optimal level for Player 1 is $x^* = 1 - (p - ps^*)/(s^* - ps^*)$, where $s^* \approx 0.3533$ and the optimal level for Player 2 is $y^* = 1 - (p - pt^*)/(t^* - pt^*)$. The value of the game

$$
v(p) = t^*(1 - 2 \ln s^* - \ln t^* - \frac{1}{s^*} e^{s^*-1} + (\frac{1}{t^*} - \frac{1}{s^*})e^{-1}) \approx 0.1571
$$

2. If $p > t^*$, the players have to modify their strategies according to the properties of $\tilde{f}(s,t)$ on $[p,1] \times [p,1]$. For $p \in (t^*, s^*)$ an equilibrium point is $(s^*, p)$ and for $p > s^*$ the conditions (5.3) are valid for $(p, p)$ (see Fig. 1).
Table 1. The solution of the game with the geometric horizon with parameter $p$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>Strategies</th>
<th>Value of the game</th>
<th>Probability of success for Player 1: $P_p$</th>
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<td>$s^*(p)$</td>
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Note: for $p \geq s^*$ $s^*(p) = t^*(p) = p$, $v(p) = p - p \ln p - e^{p-1}$, $P_p = -p \ln p$ is independent of $p$.

(2) If $t^* < p \leq s^*$, an equilibrium point is $(x^*, 0)$ and

$$v(p) = p(1 - 2 \ln s^* - \ln p - \frac{1}{s^*}e^{s^*-1} + (\frac{1}{p} - \frac{1}{s^*})e^{-1}).$$

(3) If $s^* < p \leq 1$, an equilibrium point is $(0, 0)$ and

$$v(p) = p - p \ln p - e^{p-1}.$$

For the continuous-time game from Section 4 the optimal behavior of the players can be described, based on the solution given by Proposition 1, as follows:

(1) For $\lambda$ and $\mu$ such that $\frac{\mu}{\lambda + \mu} \leq t^*$ (i.e. when the expected number of observations is big) Player 1 chooses a level $x^* = 1 - \frac{\mu}{\lambda} \frac{1-s^*}{t^*}$ and Player 2 chooses $y^* = 1 - \frac{\mu}{\lambda} \frac{1-s^*}{t^*}$. If the value of the first candidate ($\xi_1$ is a candidate if $\xi_1 \geq t^*$) is in $[y^*, x^*)$, it is taken by Player 2 and Player 1 changes his level to $z_1 = 1 - \frac{\mu}{\lambda} \frac{1-y^*}{u^*}$, where $u^* = u^*(t^*, s^*) = e^{-1}$ is given by (5.2). If the value of the first candidate is at least $x^*$, it is taken by Player 1 and Player 2 changes his level to $z_2 = 1 - \frac{\mu}{\lambda} \frac{1-u^*}{u^*}$, where $u^* = u^*(s^*, 1) = e^{s^*-1}$ (levels $z_1$ and $z_2$ do not depend on the moment of first acceptance).

(2) If $t^* < \frac{\mu}{\lambda + \mu} \leq s^*$, Players 1 and 2 choose levels $x^*$ and 0, respectively. If the first observation is less than $x^*$, it is taken by Player 2 and Player 1 changes his level to $z_1 = 1 - \frac{\mu}{\lambda} \frac{1-u^*}{u^*}$, where $u^* = u^*(\frac{\mu}{\lambda + \mu}, s^*) = e^{-1}$. If the first observation is at least $x^*$, it is taken by Player 1 and Player 2 changes his level to $z_2 = 1 - \frac{\mu}{\lambda} \frac{1-u^*}{u^*}$, where $u^* = u^*(s^*, 1) = e^{s^*-1}$. 


(3) If \( \frac{\nu}{\lambda + \mu} > s^* \), both Players choose level 0. In this case the expected number of observations is small. The first observation is always taken by Player 1 and then Player 2 changes his level to \( z_2 = 1 - \frac{\nu}{\lambda} \frac{1 - u^*}{\mu} \), where \( u^* = u^*(-\frac{\mu}{\lambda + \mu}, 1) = e^{\frac{-\lambda}{\lambda + \mu} - 1} = e^{-\frac{\lambda}{\lambda + \mu}} \).

**Figure 2.** Value and the probability of success for Player 1


(1) If \( p \geq s^* \), the minimax equations for the auxiliary game are valid for any pair \( (p, t), t \in [p, 1] \) (i.e. in main problem, Player 2 can use as an optimal strategy any level \( y \geq 0 \)). The first observation will be chosen by Player 1 (if it appears), then Player 2 chooses his new level \( u_2 = u^*(p, 1) = e^{p - 1} \) and waits for the first candidate.

(2) It is interesting and quite unexpected that in all natural situations (i.e. when \( p \) is small, i.e. \( p \leq t^* \)) the value of the game \( v(p) \) is constant (see also Figure 2 and Table 1).

(3) In formulæ (3.1) the sum of all the components associated with 't' gives the probability \( P_p \) of success for Player 1. \( P_p \) in new coordinates, when the pure strategies \( (s, t) \) and then the optimal \( u_1^*, u_2^* \) are used, has the following form for \( s \geq t \):

\[
P_p(s, t) = \begin{cases} 
    e^{-1} - t (\frac{1}{s} e^{-1} + \ln s) & \text{if } s < e^{-1}, \\
    e^{-1 + \frac{t}{1 + \ln s}} - t (1 + \ln s) & \text{if } s \geq e^{-1}.
\end{cases}
\]

It is surprising that for \( p \leq t^* \) the probability of success for Player 1, when the optimal strategies are used, \( P_p = P_p(s^*, t^*) \approx 0.3677 \) is only a little less than the probability of success in the one person model \( (e^{-1} \approx 0.3679, \text{ see Porosiński [5]} \).
References


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