MIXED-TYPE SECRETARY PROBLEMS ON SEQUENCES OF BIVARIATE RANDOM VARIABLES

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ABSTRACT. An employer interviews a finite number $n$ of applicants for a position. They are interviewed one by one sequentially in random order. As each applicant $i$ is interviewed, two attributes are evaluated by the amounts $X_i$ and $Y_i$, where $X_i$ may be "talent" (or quality), and $Y_i$ may be the "look" (or degree of favorable impression) of the applicant. Suppose that $\{X_i\}_{i=1}^n$, $\{Y_i\}_{i=1}^n$ is under the condition of full (no) - information secretary problem and that $X_i$'s and $Y_i$'s are mutually independent. We consider the three kinds of the employer's object and for each of three cases the problem is formulated by dynamic programming, and the optimal policy is explicitly derived.

1. Introduction. An employer interviews a finite number $n$ of applicants for a position. They are interviewed one by one sequentially in random order. Each applicant $i$ has two attributes $X_i$, i.e. talent, ability or quality, and $Y_i$, i.e. the look or the degree of favorable impression. Suppose that $\{X_i\}_{i=1}^n$ is an iid sequence of r.v.s with common cdf $F(x)$, and $\{Y_i\}_{i=1}^n$ is an independent sequence of r.v.s such that

$$P(Y_i = j) = \frac{1}{i}, \ j = 1, 2, \ldots, i.$$  

We assume that the two sequences $\{X_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^n$ are mutually independent. The employer observes $(X_i, Y_i)$ sequentially one by one, as each applicant appears, and he must choose (=hire or stop) one applicant without recall (i.e. an applicant is once not hired, she is rejected and cannot be recalled later). The objective of the employer is to:

(1) maximize the expected quality (i.e. $X$) of the applicant hired, with the condition that his (or her) look (i.e. $Y$) is the best among all applicants.

(2) i.i.d., with the more generous condition that the look should be the best or the second best.

(3) maximize the expected utility of the absolute rank in the look of the applicant hired, with the condition that his (or her) quality is highest among all applicants.

The purpose of the present paper is to find the optimal hiring policy to achieve each object (1)~(3), by formulating and solving the problem by dynamic programming.

Our assumption that $X_i$'s and $Y_i$'s are mutually independent is, of course, too much restrictive, and doesn't fit our real world. Looks and talent may be correlated. Dependence between the two attributes $X$ and $Y$ will introduce a new class of secretary problems different from that discussed in this paper (see [5]).

Moreover the problems we consider here belong to a mixed-type of full-information (FI) times no-information (NI) secretary problems. We consider, also NI-times-NI and FI-times-FI types of secretary problems, in the similar context as discussed in the paper. The present

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work may be an alternative approach to the secretary problem for bivariate r.v's than that which was tried in Sakaguchi [4].

A very important and now classical literature in FI and NI secretary problems is Gilbert and Mosteller [1]. Recent look for the secretary problem and its extension can be found in Samuels [6].

The above formulated tasks will be solved by the optimal stopping theory methods. Let us recall the formulation of such optimization problems. Let \((\xi_n, \mathcal{F}_n, P_x)_{n=0}^N\) be a Markov process defined on the probability space \((\mathbb{E}, \mathcal{B})\). Define \(g: \mathbb{E} \to \mathbb{R}\) such that \(E_x|g(X_1)| < \infty\) and \(\mathcal{S}\) be the set of stopping times with respect to filtration \((\mathcal{F}_n)_{n=0}^N\). The optimal stopping problem can be formulated as determining

\[
v(x) = \sup_{\tau \in \mathcal{S}} E_x g(\xi_\tau)
\]

and \(\tau^* \in \mathcal{S}\) such that \(E_x g(\xi_{\tau^*}) = v(x)\). The problems considered in this paper (1)~(3) can be reformulated as the optimal stopping problem of bidimensional Markov chain \(\xi_n = (X_i, Y_i)\) with the gain function \(g(x, y) = h_1(x)h_2(y)\). One of the coordinate is related to the no information best choice problem or its generalization. The description of the problem is as follows. Let \(K = \{x_1, x_2, \ldots, x_N\}\) be the set of characteristics, assuming that the values are different. The decision maker observes a permutation \(\{\eta_1, \eta_2, \ldots, \eta_N\}\) of the elements of \(K\) sequentially. All permutations are equally likely. Let \(R_k\) denote the absolute rank of the object with the characteristics \(\eta_k\), i.e.

\[
R_k = \min\{r : \eta_k = \bigwedge_{1 \leq i_1 < \ldots < i_r \leq N} \bigvee_{1 \leq j \leq r} \eta_{i_j}\},
\]

(\(\wedge\) and \(\vee\) denote minimum and maximum, respectively). The decisions at each time are based on relative ranks \(Y_1, Y_2, \ldots, Y_N\) of the applicants, where

\[
Y_k = \min\{r : \eta_k = \bigwedge_{1 \leq i_1 < \ldots < i_r \leq k} \bigvee_{1 \leq j \leq r} \eta_{i_j}\}.
\]

(1.1)

One can define the auxiliary payoff function

\[
g_a(r, l) = P\{R_r = a | Y_r = l\} = \frac{(a-1)^{N-a}}{(N-1)} \frac{N-a}{r},
\]

(1.2)

where \(a = 1, 2, \ldots, N; l = 1, 2, \ldots, \min(a, r); r = 1, 2, \ldots, N\). The argument of the second coordinate is bidimensional. The first argument is the number of observation and the second one is the relative rank. The parameter \(a\) is the absolute rank of observation.

Let \(c_{R_i}\) be the payoff when we choose the object with absolute rank \(R_i\). If we are using the strategies based on knowledge of the relative ranks then we have to construct the payoff measurable with respect to suitable filtration. Let \(\xi_i = (X_i, Y_i)\), where \(Y_i\) is defined by (1.1). Denote \(\mathcal{F}_i = \sigma(\xi_1, \xi_2, \ldots, \xi_i)\) and let \(\tau \in \mathcal{S}\). We have

\[
E_{c_{R_\tau}} = \sum_{i=1}^n \int_{\{\tau = i\}} c_{R_i} dP = \sum_{i=1}^n E[c_{R_i}|\mathcal{F}_i]dP
\]

\[
= \sum_{i=1}^n \int_{\{\tau = i\}} E[c_{R_i}|Y_i]dP = \sum_{i=1}^n \int_{\tau = i} \varphi(i, Y_i) dP
\]

\[
= E\varphi(\tau, Y_\tau),
\]

(1.3)
where

\[ \varphi(t, y) = \sum_{r=y}^{n} g_r(t, y)c_r. \tag{1.4} \]

Since the part of the gain function is related to the payoff in the no information secretary problem, it is convenient to consider the Markov chain with state space on which the gain function is strictly positive. Let us assume that the gain function is 0 for \( l > \min(a, r) \) and positive for \( l \leq \min(a, r) \). It means the we are looking for the states \((r, l)\) such that \( l \leq \min(a, r) \).

Let \( W_1 = (1, Y_1) = (1, 1) \) and for \( t > 1 \) let \( \gamma_t = \inf\{r > \gamma_{t-1} : Y_r \leq \min(a, r)\} \), \((\inf \emptyset = \infty)\) and \( W_t = (\gamma_t, Y_{\gamma_t}) \). If \( \gamma_t = \infty \), then we put \( W_t = (\infty, \infty) \). The process \( W_t \) is the Markov chain with the following one step transition probabilities

\[ p(r, s) = P\{W_{t+1} = (s, l_s)|W_t = (r, l_r)\} = \begin{cases} \frac{1}{s} & \text{for } r < a, \ s = r + 1, \\ \frac{(s)_a}{(s)_a + 1} & \text{for } a \leq r < s, \\ 0 & \text{for } r \geq s \text{ or } a < s, \ s \neq r + 1, \end{cases} \tag{1.5} \]

\[ p(\infty, \infty) = 1, \ p(r, \infty) = 1 - a \sum_{s=r+1}^{N} p(r, s), \tag{1.6} \]

where \((s)_a = s(s-1)(s-2) \ldots (s-a+1); (s)_0 = 1\). Denote \( \mathcal{G}_t = \sigma(X_{\gamma_1}, W_1, X_{\gamma_2}, W_2, \ldots, X_{\gamma_t}, W_t) \) and \( \mathfrak{M} \) the set of the stopping times with respect to the filtration \( \{\mathcal{G}_t\}_{t=1}^{\infty} \).

2. Selecting good quality together with the best look. Let \((X_i, Y_i)_{i=1}^{n} \) be a sequence of independent bivariate r.v.s as given in the previous section. Observing the sequence \((X_i, Y_i), i = 1, 2, \ldots, n\), one by one sequentially, we want to maximize \( EX_{\tau}c_{R_{\tau}} \), where \( \tau \) is the stopping time belonging to \( \mathcal{S} \) and \( c_1 = 1 \) and \( c_i = 0 \) for \( i = 2, 3, \ldots, n \), i.e., we have \( a = 1 \) in (1.2) - (1.6). We have then by (1.3) and (1.4), for fixed horizon \( n \), that for \( \tau \in \mathcal{S} \) there is \( \sigma \in \mathfrak{M} \) such that

\[ EX_{\tau}c_{R_{\tau}} = Eg(X_{\gamma_\sigma}, W_\sigma) = Eg(X_\sigma, \gamma_\sigma). \tag{2.1} \]

We define state \((x, i)\) which means that the \( i \)-th object has the first attribute \( X_i = x \) and the second attribute \( Y_i = 1 \). The horizon is \( n \) and the stop reward at state \((x, i)\) is, therefore,

\[ g(x, i) = \frac{i}{n}. \]

Denoting, by \( v_n(x, i) \), the expected reward obtained by employing the optimal stopping rule for the \( n \)-object problem at state \((x, i)\), we easily have, by (1.6) that

\[ P\{Y_{i+1}, \ldots, Y_{j-1} \geq 2 \text{ and } Y_j = 1|Y_i = 1\} = p(i, j) = \frac{i}{j(j-1)}, \]

and

\[ v_n(x, i) = \max\{ \frac{i}{n}, \sum_{j=i+1}^{n} \frac{i}{j(j-1)} E_{F} v_n(X, j) \}, \tag{2.2} \]
\( i = 1, 2, \ldots, n; \, v_n(x, u) \equiv x \). Letting \( V_{n, i} \equiv \frac{n}{i} E_F v_n(X, i) \), and \( d_{n, i} \equiv \sum_{j=i+1}^{n} \frac{v_{n,j}}{j-1} \), by equation (2.2) we have

\[
V_{n, i} = E_F (X \vee d_{n, i}) = S_F (d_{n, i}),
\]

where \( S_F (t) \) is defined by \( S_F (t) = E_F (X \vee t) \).

**Theorem 1.** The optimal stopping rule for the optimality equation (2.2) is: "Stop at the earliest object \((X_i, Y_i)\) whose relative rank \(Y_i\), when it appears, is unity and satisfies \(X_i > d_{n, i}\)." The sequence \( \{d_{n, i}\}_{i=1}^{n} \) is determined by the recursion

\[
d_{n, i} = \frac{S_F (d_{n, i+1})}{i} + d_{n, i+1},
\]

\( i = 1, 2, \ldots, n - 1; \, d_{n, n} = 0. \)

The optimal expected reward is given by \( E_F v_n(X, 1) \), i.e. \( S_F (d_{n, 1})/n \).

**Proof.** The recursion (2.4) follows from the definition of \( d_{n, i} \) and (2.3). \( \square \)

**Example 1.** Let \( \{X_i\}_{i=1}^{n} \) be i.i.d. sequence of r.v.s with the uniform distribution on \([0, 1]\) i.e. common cdf has a form \( F(x) = x \) \((0 \leq x \leq 1)\) and \( n = 10 \). Then, since \( S_F (t) = (1 + t^2)/2 \) for \( 0 \leq t \leq 1 \); and \( = t \) for \( t > 1 \), we have

\[
S_F (d_{10, i}) = \begin{cases} 
\frac{1 + d_{10,i}^2}{2} & \text{if } 0 \leq d_{10,i} \leq 1 \\
d_{10,i} & \text{if } d_{10,i} > 1.
\end{cases}
\]

Hence, from (2.4), we get Table 1 which shows that the optimal rule is given by "Pass the first two objects and stop at the earliest object \((X_i, Y_i)\), \(3 \leq i \leq 10\), that satisfies \(Y_i = 1\) and \(X_i > d_{10,i}\)." The expected reward obtained by employing this rule is \( \frac{S_F (d_{10, 1})}{10} \approx 0.2258 \).

**Table 1.** Decision points and values of the problem

<table>
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<tr>
<th>( i )</th>
<th>( d_{10,i} )</th>
<th>( S_F (d_{10,i}) )</th>
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**Remark 1.** Consider the following two non-optimal stopping problems. The first one is to attempt to maximize \( E_F X_r \) and leave \( Y_s \)’s in a random. This rule gives the expected reward \( \frac{1}{n}v_n \), where \( v_n \) is given by Moser’s sequence \( v_n = (1 + v_{n-1}^2)/2 \), \((n \geq 1, \, v_0 = 0)\)(see Remark 2 on p.105). The second one is to attempt to maximize the probability of selecting the best look and leave \( X_s \)’s in a random manner (see [1] Section 2a and Table 2). The optimal expected reward \( \frac{1}{n}S_F (d_{n, 1}) \) in Theorem 1 is not smaller than \( \max \{ \frac{1}{n}v_n, E_F (X)\pi (s_n^n, n) \}. \) We find that this is true since \( \frac{1}{10}S_F (d_{10, 1}) \approx 0.2258, \, v_{10} \approx 0.8611 \) and \( \pi (s_{10}^n, 10) \approx 0.3987. \)
3. Selecting good quality together with one of the two bests in the look. We consider the same problem as in the previous section with a single difference that the required condition at the stopping time $\tau \in \mathfrak{S}$ is taken a little bit more generously. Let us denote by $R_i$, as in Section 2, the absolute rank of the $i$-th object in the second attribute and let $c_1 = 1$, for $i = 1, 2$ and $c_i = 0$, for $i = 3, 4, \ldots, n$. We can use formulæ (1.2) - (1.6) with $a = 2$. The problem is to maximize $EX_\tau c_{R_\tau}$, where $\tau \in \mathfrak{S}$. We have that for $\tau \in \mathfrak{S}$ there is $\sigma \in \mathfrak{M}$ such that

$$EX_\tau c_{R_\tau} = P(R_\tau = 1 \text{ or } 2)E[X_\tau | R_\tau = 1 \text{ or } 2] = E\tilde{g}(X_{\gamma_\sigma}, W_\sigma) = Eg(X_{\gamma_\sigma}, \gamma_\sigma, Y_{\gamma_\sigma}).$$

When horizon is $n$ we are in state $(x, i, k)$ when $X_i = x$ and the second attribute has relative rank $Y_i = k$. The stop reward is

$$g(x, i, k) = \begin{cases} 
    P(R_i = 1 \text{ or } 2 | Y_i = 1)x = \frac{i(2n_i - i - 1)x}{n(n-1)}, & \text{for } k = 1, \\
    P(R_i = 2 | Y_i = 2)x = \frac{i(i - 1)x}{n(n-1)}, & \text{for } k = 2,
\end{cases}$$

which follows from (1.2).

Denote, by $v_n(x, i)$, the expected reward obtained by using the optimal stopping rule for the $n$-period problem at state $(x, i, 1)$ and denote by $u_n(x)$, the similar one at the state $(x, i, 2)$. We find that

$$v_n(x, i) = \max \left\{ \frac{i(2n_i - i - 1)}{n(n-1)}x, \Phi_n(i) \right\},$$

and

$$u_n(x, i) = \max \left\{ \frac{i(i - 1)}{n(n-1)}x, \Phi_n(i) \right\},$$

where

$$\Phi_n(i) = \sum_{j=i+1}^{n} \frac{i(i - 1)}{j(j - 1)(j - 2)}E_F\{v_n(X, j) + u_n(X, j)\}$$

$(i = 1, 2, \ldots, n; v_n(x, n) = u_n(x, n) = x)$ since we have by (1.2) - (1.6) with $a = 2$

$$P\{Y_{i+1} \ldots Y_{j-1} \geq 3 \text{ and } Y_j \leq 2 | Y_i = k\} = p(i, j) = \frac{i(i - 1)}{j(j - 1)(j - 2)},$$

for $k = 1, 2$. Letting $V_{n, i} = \frac{n}{i}E_Fv_n(X, i)$ and $U_{n, i} = \frac{n}{i}E_Fu_n(X, i)$. Equations (3.1)-(3.2)

becomes

$$V_{n, i} = \frac{2n - i - 1}{n - 1}S_F(d_{n, i}),$$

$$U_{n, i} = \frac{i - 1}{n - 1}S_F(e_{n, i}),$$

$(i = 1, 2, \ldots, n; U_{n, n} = V_{n, n} = E_F(X))$, where

$$d_{n, i} = \frac{n - 1}{2n - i - 1} \sum_{j=i+1}^{n} \frac{i - 1}{(j - 1)(j - 2)}(V_{n, j} + U_{n, j}),$$

and

$$e_{n, i} = \frac{2n - i - 1}{i - 1}d_{n, i}.$$
Thus we can prove

**Theorem 2.** The optimal stopping rule for the optimality equations (3.1)-(3.2) is: "Stop at either the earliest state \((X_i, i, Y_i)\) satisfying \(Y_i = 1\) and \(X_i > \text{d}_{n,i}\) or \(Y_i = 2\) and \(X_i > \text{e}_{n,i}\), whichever occurs first". The sequence \(\{\text{d}_{n,i}\}\) and \(\{\text{e}_{n,i}\}\) are determined by the recursion

\[
\text{d}_{n,i} = \frac{1}{(2n - i - 1)}[(n - 1)(\text{V}_{n,i+1} + \text{U}_{n,i+1}) + (2n - i - 2)(i - 1)\text{d}_{n,i+1}],
\]

\((i = 1, 2, \ldots, n - 1; \text{V}_{n,n} = \text{U}_{n,n} = E_F(X), \text{d}_{n,n} = \text{e}_{n,n} = 0)\), where \(\{\text{V}_{n,i}\}, \{\text{U}_{n,i}\}\) and \(\{\text{e}_{n,i}\}\) satisfy (3.3)-(3.4) and (3.6). The optimal expected reward is given by \(E_Fv_n(X, 1)\), i.e. \(\frac{2}{n}S_F(d_{n,1})\).

**Proof.** The recursion (3.7) follows from (3.5). The rest is the consequence of easy calculations. □

**Example 2.** Let \(F(x) = x\), \((0 \leq x \leq 1)\) and \(n = 10\). Since we have, from (3.7) and (3.6),

\[
\begin{align*}
\text{d}_{10,i} &= \frac{1}{(19 - i)^2} [9(\text{V}_{10,i+1} + \text{U}_{10,i+1}) + (18 - i)(i - 1)\text{d}_{10,i+1}] \\
\text{e}_{10,i} &= \frac{19 - i}{i - 1} \text{d}_{10,i},
\end{align*}
\]

where

\[
\begin{align*}
\text{V}_{10,i} &= \frac{19 - i}{9} [\frac{1}{2}(1 + d_{10,i}^2) \text{ or } d_{10,i}], \\
\text{U}_{10,i} &= \frac{i - 1}{9} [\frac{1}{2}(1 + e_{10,i}^2) \text{ or } e_{10,i}],
\end{align*}
\]

we get a table

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<th>(i)</th>
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<th>(V_{10,i})</th>
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This table shows that the optimal rule is: "Pass the first object and stop at either the earliest state \((X_i, i, Y_i)\), \(2 \leq i \leq 9\), satisfying \(Y_i = 1\) and \(X_i > d_{10,i}\) or \(6 \leq i \leq 9\), satisfying \(Y_i = 2\) and \(X_i > e_{10,i}\), whichever occurs first. If none of the above event occurs, stop at \((X_{10}, Y_{10})\)." The expected reward obtained by employing this rule is \(\frac{2}{10}S_F(d_{10,1}) = 0.3731\).
Remark 2. If the interviewer considers the second component $Y_i$ as unimportant and neglects it, and wants to select one with $X_i$ as large as possible, then the optimal stopping rule is: "Stop at the earliest one with $X_i > x_{n-i}$, where the sequence $\{v_m\}_{m=1}^n$ is determined by $v_m = S_p(v_{m-1})$, $m = 1, 2, \ldots; v_0 \equiv 0$, and the optimal expected reward is equal to $v_n$. We know, for example, that if $F(x) = x$, $0 \leq x \leq 1$, then $v_m = \frac{1 + v_{m-1}}{2}$ (i.e., Moser's sequence and $v_{10} = 0.8611$, see [1], Table 11 in Section 5). Also for this case Example 1 (2) shows that the requirement that the selected object should be the best (the best or the second best) in the look diminishes the optimal value down to 0.2258 (0.3731).

4. Selecting good look together with the best quality. Let $\{(X_i, Y_i)\}_{i=1}^n$ be an iid sequence of r.v.s as in Section 1 and assume that $\{X_i\}_{i=1}^n$ has uniform distribution over the unit interval $0 \leq x \leq 1$. Observing the sequence $\{(X_i, Y_i)\}_{i=1}^n$, one by one sequentially, we want to maximize $E_{C_{R_i}} \chi_{\{X_r = \max_{1 \leq i \leq n} X_i\}}$, where the stopping time $\tau \in \mathcal{G}$. One can reformulate the problem as the optimal stopping problem for some Markov chain. Denote $\xi_\tau = \chi_{\{X_r = \max_{1 \leq i \leq n} X_i\}}$. We have

\begin{equation}
E_{C_{R_i}} \chi_{\{X_r = \max_{1 \leq i \leq n} X_i\}} = \sum_{s=1}^n \int_{\{\tau = s\}} c_{R_i} \chi_{\{X_r = \max_{1 \leq i \leq n} X_i\}} dP
\end{equation}

\begin{align*}
&= \sum_{s=1}^n \int_{\{\tau = s\}} E[c_{R_i} \chi_{\{X_r = \max_{1 \leq i \leq n} X_i\}} | \mathcal{F}_s] dP \\
&= \sum_{s=1}^n \int_{\{\tau = s\}} E[c_{R_i} \chi_{\{X_r = \max_{1 \leq i \leq n} X_i\}} | \xi_s, Y_s] dP \\
&= \sum_{s=1}^n \int_{\{\tau = s\}} \psi(s, \xi_s, Y_s) dP = E_{\tau, Y_\tau} = E_{\psi}(\tau, \xi_\tau, Y_\tau).
\end{align*}

Let $Z_1 = (1, X_1)$ and for $t > 1$ let $\gamma_t = \inf\{r > \gamma_{t-1} : \xi_r = 1\}$, $(\inf \emptyset = \infty)$ and $Z_t = (\gamma_t, X_{\gamma_t})$. If $\gamma_t = \infty$, then we put $Z_t = (\infty, \infty)$. The process $Z_t$ is the Markov chain with the following one step density of transition probabilities

\begin{equation}
p(i, x, (j, z)) = \begin{cases} 
x^{j-i-1}, & \text{for } j > i, x < z \\
0, & \text{otherwise.}
\end{cases}
\end{equation}

For calculation of (4.1) it is enough to observe the sequence $\{(Z_t, Y_{\gamma_t})\}_{t=1}^n$. Let $\mathcal{G}_t = \sigma(Z_1, Y_1, Z_2, Y_2, \ldots, Z_t, Y_t)$ and $\mathfrak{M}$ the set of stopping times with respect to $\{\mathcal{G}_t\}_{t=1}^n$. If we face the $\gamma_t = i$-th object $(X_r, Y_r)$ with $X_i = \max(X_1, X_2, \ldots, X_n) = x$ and $Y_i = k$ we have Markov chain $\{\xi_{\gamma_t}, X_{\gamma_t}, Y_{\gamma_t}\}_{t=1}^n$ in state $(i, x, k)$. The stop reward at state $(i, x, k)$ is, based on the above consideration and from (1.2)

\begin{equation}
g(i, x, k) = \frac{x^{n-i}}{\binom{n}{i}} \sum_{r=k}^{n} \binom{r-1}{k-1} \binom{n-r}{i-k} c_r,
\end{equation}

for $1 \leq k \leq i, i \leq r \leq n, 1 \leq i \leq n$ and $0 \leq x \leq 1$. We assume that $c_1 \geq c_2 \geq \ldots \geq c_n \geq 0$. The gain function $g(i, x, k)$ is defined on $\{1, 2, \ldots, n\} \times [0, 1] \times \{1, 2, \ldots, n\}$. For every $\tau \in \mathcal{G}$ we have $\sigma \in \mathfrak{M}$ such that

\begin{equation}
E_{C_{R_i}} \chi_{\{X_r = \max_{1 \leq i \leq n} X_i\}} = E_{\psi}(\tau, \xi_\tau, Y_\tau) = E_{g}(\gamma_\sigma, X_{\gamma_\sigma}, Y_{\gamma_\sigma}).
\end{equation}
Define

\begin{equation}
T_g(i, x, k) = E(i, x, k)g(\gamma, X_\gamma, Y_\gamma)
\end{equation}

\begin{equation}
= \sum_{r=i+1}^{n} \sum_{u=1}^{n} \frac{1}{r} \int_{x}^{1} \frac{z^{n-r}}{r} \sum_{s=u}^{n} \frac{(s-1)(n-s)}{(r-u)} \frac{x^{-i-1}}{c_s} dz
\end{equation}

\begin{equation}
= \sum_{r=i+1}^{n} \frac{x^{r-i-1} - x^{n-i}}{r(n-r+1)} \sum_{u=1}^{n} \frac{(s-1)(n-s)}{(r-u)} \frac{1}{c_s}
\end{equation}

Denoting, by \(v_n(i, x, k)\), the expected reward obtained by employing the optimal stopping rule for the \(n\)-period problem at state \((i, x, k)\), we easily have

\begin{equation}
v_n(i, x, k) = \max \left[ \frac{x^{n-i}}{(n)} \sum_{r=k}^{n} \frac{(r-1)(n-r)}{(k-1)(i-k)} c_r, \right.
\end{equation}

\begin{equation}
\sum_{j=i+1}^{n} \frac{x^{j-i-1}}{j} \sum_{w=1}^{j} \int_{x}^{1} v_n(j, z, w) dz \right],
\end{equation}

where \(1 \leq k \leq i, 1 \leq i \leq n, 0 \leq x \leq 1\), with \(v_n(n, x, k) = c_k\).

The one-step look ahead stopping region ([2]; pp. 137-139) corresponding to this optimality equation is

\begin{equation}
B = \{(i, x, k)|g(i, x, k) \geq T_g(i, x, k)\}
\end{equation}

\begin{equation}
= \{(i, x, k)|\sum_{r=k}^{n} \frac{(r-1)(n-r)}{(k-1)(i-k)} c_r, \right.
\end{equation}

\begin{equation}
\sum_{j=i+1}^{n} \left[ \frac{x^{-n+j-1} - 1}{j(n-j+1)} \sum_{w=1}^{n} \frac{(r-1)(n-r)}{(w-1)(j-w)} a_r \right].
\end{equation}

We discuss the following two simple cases.

\textbf{Case 1.} \(a_1 = 1, a_2 = a_3 = \ldots = a_n = 0\).

For this case the stop reward \((4.3)\) becomes

\[ g(i, x, k|n) = \begin{cases} \frac{ix^{n-i}}{n}, & \text{if } k = 1 \\ 0, & \text{if } 2 \leq k \leq i \end{cases} \]

and the one-step stopping region \((4.6)\) becomes, after simplification,

\begin{equation}
B = \{(i, x, k|n)|1 \geq \frac{1}{i} \sum_{l=1}^{n-i} \frac{x^{-l-1}}{l}\}
\end{equation}

\textbf{Lemma 1.} The region \(B\) given by (4.7) is "closed", i.e., if once a state enters \(B\), the state never leaves \(B\) as the process goes on.
Proof. For any $0 \leq x < z \leq 1$ we have

$$i^{-1} \sum_{l=1}^{n-i} \frac{x^{-l} - 1}{l} \geq i^{-1} \sum_{l=1}^{n-i} \frac{z^{-l} - 1}{l}$$

$$= i^{-1} \left[ \sum_{l=1}^{n-i-1} \frac{z^{-l} - 1}{l} + \frac{z^{-(n-i)} - 1}{n-i} \right] \geq \frac{1}{i+1} \sum_{l=1}^{n-i-1} \frac{z^{-l} - 1}{l}.$$  

Hence

$$1 \geq \frac{1}{i} \sum_{l=1}^{n-i} \frac{x^{-l} - 1}{l} \Rightarrow 1 \geq \frac{1}{i+1} \sum_{l=1}^{n-i-1} \frac{x^{-l} - 1}{l}. \quad \Box$$

Let $d_{n,i}$ ($i = 1, 2, \ldots, n - 1; d_{n,n-1} = n^{-1}$, $d_{n,n} = 0$) be a unique root in $[0, 1]$ of the equation

$$(4.8) \quad \frac{1}{i} \sum_{l=1}^{n-i} \frac{x^{-l} - 1}{l} = 1.$$ 

Evidently

$$1 \geq \frac{1}{i} \sum_{l=1}^{n-i} \frac{x^{-l} - 1}{l} \iff x > d_{n,i}.$$ 

For some small $n$, we find that $d_{2,1} = 1/2$, $d_{3,2} = 1/3$, $d_{3,1} = \frac{1 + \sqrt{8}}{5} \approx 0.6899$, $d_{4,1} = 1/4$, $\frac{1 + \sqrt{8}}{7} \approx 0.5469$, $0.7755$ for $i = 3, 2, 1$, respectively. $d_{5,1} = 1/5$, $\frac{1 + \sqrt{10}}{9} \approx 0.4625$, $0.6591$, $0.8246$ for $i = 4, 3, 2, 1$, respectively and so on.

It is well known that if the one-step stopping region is realizable and “closed”, it becomes the optimal stopping region. From Lemma 1 we thus have

**Theorem 3.** The optimal stopping region for the optimality equation (4.5) in Case 1, is:

"stop at the earliest $(X_i, Y_i)$ that satisfies $Y_i = 1$ and $X_i = \max_{1 \leq t \leq i} X_t > d_{n,i}$, where each $d_{n,i}$ is given by a unique root of the equation (4.8)."

**Example 3.** For $n = 5$ the optimal stopping rule is as follows.
\[
\begin{align*}
\text{If } & \quad \begin{cases} 
y_1 = 1 & \text{if } x_1 \geq 0.8246 \\
\text{otherwise} & 
\end{cases}, \\
\text{then } & \quad \begin{cases} 
\text{stop,} \\
\text{observe } (x_2, y_2) 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{If } & \quad \begin{cases} 
y_2 = 1 & \text{if } x_2 \geq x_1 \lor 0.6591 \\
\text{otherwise} & 
\end{cases}, \\
\text{then } & \quad \begin{cases} 
\text{stop,} \\
\text{observe } (x_3, y_3) 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{If } & \quad \begin{cases} 
y_3 = 1 & \text{if } x_3 \geq x_1 \lor x_2 \lor 0.4625 \\
\text{otherwise} & 
\end{cases}, \\
\text{then } & \quad \begin{cases} 
\text{stop,} \\
\text{observe } (x_4, y_4) 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{If } & \quad \begin{cases} 
y_4 = 1 & \text{if } x_4 \geq x_1 \lor x_2 \lor x_3 \lor 0.2 \\
\text{otherwise} & 
\end{cases}, \\
\text{then } & \quad \begin{cases} 
\text{stop,} \\
\text{observe } (x_5, y_5) \lor \text{stop.} 
\end{cases}
\end{align*}
\]

**Remark 3.** Suppose that the interviewer does not observe the second component \(Y_i\) or does observe but without considering it as important, and wants to only maximize the probability of selecting the applicant with the highest \(X_i\). Then the well-known result (see [1], Table 8 in Section 3) in the theory of the secretary problem is that the optimal stopping rule is: “Stop at the earliest applicant with \(X_i = \max_{1 \leq t \leq i} X_t > v_{n-i}\), where each \(v_m\) in the sequence \(\{v_m\}_{m=1}^n\) is determined by

\[
(4.9) \quad \sum_{j=1}^{m} \frac{v^{-j} - 1}{j} = 1, \quad (m = 1, 2, \ldots).
\]

We know that \(v_i\) = 0.8246, 0.7758, 0.6899, 0.6392 1/2, for \(i = 5, 4, 3, 2, 1\), respectively, and compare these values with \(d_{5, i}\)'s in Example 3.

**Remark 4.** (Continuation to Remark 2). In Remark 2 it is known that the probability of selecting the maximum of \(X\) by following the optimal strategy is

\[
(4.10) \quad P_n = \frac{1}{n} \left[ 1 + \sum_{i=1}^{n-1} \sum_{y=1}^{i} \frac{v_{i-y}}{n-y} \right]
\]

(see [1, Section 3] and [3]), and hence

\[
P_5 = \frac{1}{5} \left[ 1 + \sum_{i=1}^{4} \sum_{y=1}^{i} \frac{v_{i-y}}{5-y} \right] = \frac{1}{5} \left[ 1 + \frac{1}{4} v_1^4 + \left( \frac{1}{4} v_2^4 + \frac{1}{3} v_2^3 \right) \right.
\]

\[
+ \left( \frac{1}{4} v_3^4 + \frac{1}{3} v_3^3 + \frac{1}{2} v_3^2 \right) + \left( \frac{1}{4} v_4^4 + \frac{1}{3} v_4^3 + \frac{1}{2} v_4^2 + v_4 \right) \right] = 0.6392
\]

where \(v_i\)’s are given by (4.9).
The expected reward obtained by employing the optimal strategy in Theorem 3 is \( W_n \equiv \int_0^1 v_n(1, x, 1) \, dx \), and the explicit expression of \( W_n \), like (4.10) is presently not known. It is clear, however, that by solving optimality equation (4.5) for Case 1, i.e.

\[
v_n(i, x) = \max \left[ \frac{i}{n} x^{-i}, \sum_{j=i+1}^{n} \frac{x^{j-i-1}}{j} \int_x^1 v_n(j, z) \, dz \right]
\]

\((i = 1, 2, \ldots, n; v_n(n, x) \equiv 1; 0 \leq x \leq 1)\), (we need not consider states in which \( 2 \leq y_i \leq i \)) recursively downward, we can derive \( v_n(1, x) \) and hence obtain \( W_n \). But it is a tediously lengthy job. For instance, we find, for \( n = 5 \)

\[
v_5(5, x) = 1
\]

\[
v_5(4, x) = \frac{4}{5} x \sqrt{\frac{1}{5}} \int_x^1 v_5(5, z) \, dz = \left( \frac{4}{5} x \right) \sqrt{\frac{1-x}{5}},
\]

\[
v_5(3, x) = \left( \frac{3}{5} x^2 \right) \sqrt{\frac{1}{5}} \sum_{j=4}^{5} \frac{x^{j-4}}{j} \int_x^1 v_5(j, z) \, dz
\]

\[= \left( \frac{3}{5} x^2 \right) \sqrt{\frac{1}{5}} \Phi_5(3, x),
\]

where

\[
\Phi_5(3, x) = \left\{ \begin{array}{ll}
\frac{21}{200} + \frac{3}{20} x - \frac{7}{40} x^2, & \text{if } 0 \leq x \leq \frac{1}{5} \\
\frac{1}{10} + \frac{3}{10} x - \frac{3}{10} x^2, & \text{if } \frac{1}{5} \leq x \leq 1 \\
\frac{3}{5} x^2, & \text{if } d_{5,3} \leq x \leq 1
\end{array} \right.
\]

and

\[
v_5(2, x) = \left( \frac{2}{5} x^3 \right) \sqrt{\frac{1}{5}} \int_x^1 \left\{ \frac{1}{3} v_5(3, z) + \frac{x}{4} v_5(4, z) + \frac{x^2}{5} v_5(5, z) \right\} \, dz
\]

and so on. Note that the unique root of \( \frac{3}{5} x^2 = \Phi_5(3, x) \) coincides with \( d_{5,3} = (1 + \sqrt{10})/9 \) derived from (4.8).

**Case 2.** \( a_1 = a_2 = 1, a_3 = a_4 = \cdots = a_n = 0. \)

For this case the optimality equation (4.5) becomes

\[
\begin{align*}
(4.11) \quad v_n(i, x) &= \max \left\{ \frac{i(2n-i-1)}{n(n-1)} x^{-i}, \Psi_n(i, x) \right\} \quad \text{for states } (i, x, 1) \\
u_n(i, x) &= \max \left\{ \frac{i(i-1)}{n(n-1)} x^{-i}, \Psi_n(i, x) \right\} \quad \text{for states } (i, x, 2).
\end{align*}
\]

where

\[
\Psi_n(i, x) = \sum_{j=i+1}^{n} \frac{x^{j-i-1}}{j} \int_x^1 (v_n(j, z) + u_n(j, z)) \, dz,
\]

and the one-step stopping region (4.6) becomes, after simplification, \( B_1 \cup B_2 \), where

\[
(4.12) \quad B_1 &= \{(i, x, 1) | 1 \geq \frac{n-1}{i(n-i)} \sum_{l=1}^{n-i} x^{-l} - 1 \} \\
B_2 &= \{(i, x, 2) | 1 \geq \frac{2(n-1)}{i(i-1)} \sum_{l=1}^{n-i} x^{-l} - 1 \}.
\]

**Lemma 2.** The region \( B_1 \cup B_2 \) is "closed".
Proof. For any $0 \leq x < z \leq 1$ we can easily prove that

$$\frac{1}{i(n - \frac{i+1}{2})} \sum_{l=1}^{n-i} \frac{x^{-l} - 1}{l} \geq \frac{1}{(i+1)(n - \frac{i+2}{2})} \sum_{l=1}^{n-i-1} \frac{z^{-l} - 1}{l},$$

and

$$\frac{1}{i(i-1)} \sum_{l=1}^{n-i} \frac{x^{-l} - 1}{l} \geq \frac{1}{i(i+1)} \sum_{l=1}^{n-i-1} \frac{z^{-l} - 1}{l}.$$

Therefore both of $B_1$ and $B_2$, and hence $B_1 \cup B_2$, are “closed”. □

Let $e_{n,i}$ ($1 \leq i \leq n$; $e_{n,n-1} = \frac{2}{n+2}$, $e_{n,n} \equiv 0$) and $f_{n,i}$ ($2 \leq i \leq n$; $f_{n,n-1} = \frac{2}{n}$, $f_{n,n} \equiv 0$) be a unique root of the equation

$$(4.13) \quad \frac{n-1}{i(n - \frac{i+1}{2})} \sum_{l=1}^{n-i} \frac{x^{-l} - 1}{l} = 1$$

and

$$(4.14) \quad \frac{2(n-1)}{i(i-1)} \sum_{l=1}^{n-i} \frac{x^{-l} - 1}{l} = 1,$$

respectively.

It is clear that $(i, x, 1) \in B_1 \iff x > e_{n,i}$ and $(i, x, 2) \in B_2 \iff x > f_{n,i}$.

For some small $n$, we find that

$$e_{2,1} = \frac{1}{2}, \quad e_{3,2} = \frac{7}{6}, \quad e_{3,3} = \frac{1+\sqrt{6}}{3} \approx 0.6899$$

$$e_{4,i} = \frac{1}{3}, \quad 3 + \frac{\sqrt{66}}{19} \approx 0.5855, 0.7755 \text{ for } i = 3, 2, 1, \text{ respectively}$$

$$e_{5,i} = \frac{2}{7}, \quad \frac{2+\sqrt{34}}{5} \approx 0.5221, 0.6834, 0.8248 \text{ for } i = 4, 3, 2, 1, \text{ respectively}$$

and

$$f_{3,2} = \frac{2}{3},$$

$$f_{4,i} = \frac{1}{2}, \quad \frac{3+\sqrt{12}}{7} \approx 0.8619, \text{ for } i = 3, 2 \text{ respectively}$$

$$f_{5,i} = \frac{2}{5}, \quad \frac{2+\sqrt{22}}{9} \approx 0.7434, 0.9246, \text{ for } i = 4, 3, 2, \text{ respectively}.$$

From Lemma 2 we obtain

**Theorem 4.** The optimal stopping region for the optimal equation (4.5) in Case 2, is: "stop at either the earliest $(X_i, Y_i)$, with $Y_i = 1$ and $X_i = \max_{1 \leq l \leq i} X_l > e_{n,i}$, or the earliest $(X_i, Y_i)$, with $Y_i = 2$ and $X_i = \max_{1 \leq l \leq i} X_l > f_{n,i}$, whichever occurs first, where \{e_{n,i}\} and \{f_{n,i}\} are defined by the unique roots of the equations (4.13)-(4.14)."

**Example 4.** For $n = 5$ the optimal stopping rule is as follows.
If \( y_1 = 1 \& x_1 \geq 0.8248 \), then \{ stop, observe \( (x_2, y_2) \) \\
If \( y_2 = 1 \& x_2 \geq x_1 \lor 0.6834 \), then \( y_2 = 2 \& x_2 \geq x_1 \lor 0.9246 \), then \{ stop, stop, observe \( (x_3, y_3) \) \\
If \( y_3 = 1 \& x_3 \geq x_1 \lor x_2 \lor 0.5221 \), then \( y_3 = 2 \& x_3 \geq x_1 \lor x_2 \lor 0.7434 \), then \{ stop, stop, observe \( (x_4, y_4) \) \\
If \( y_4 = 1 \& x_4 \geq x_1 \lor x_2 \lor x_3 \lor \frac{2}{5} \), then \( y_4 = 2 \& x_4 \geq x_1 \lor x_2 \lor x_3 \lor \frac{2}{5} \), then \{ stop, stop, observe \( (x_5, y_5) \) & stop. \\

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