Multiple Choice Problems Related to the Duration of the Secretary Problem

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Abstract

A version of the multiple choice secretary problem called the multiple choice duration problem, in which the objective is to maximize the time of possession of relatively best object, is treated. For the $m$ choice duration problem with a known number of objects, there exists a sequence of critical numbers $(s_1, s_2, \ldots, s_m)$ such that, whenever there remain $k$ choices yet to be made, then the optimal strategy immediately selects a relatively best object if it appears after or on time $s_k$, $1 \leq k \leq m$. A simple recursive formula for calculating the critical numbers when the number of objects tends to infinity will be given. It can be shown that the multiple choice duration problem with a known number of objects is related to multiple choice (best-choice) secretary problem with unknown number of objects having a uniform prior on the actual number of objects. Extensions to models involving an acquisition cost or a replacement cost are made.

Keywords: Optimal stopping; Relative ranks; Best-choice problem; Dynamic programming

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1 Introduction and summary

Ferguson, Hardwick and Tamaki [2] are the first to consider a sequential and selection problem called duration problem, which is a variation of the classical secretary problem, treated for example, by Gilbert and Mosteller [3]. The basic form of the duration problem is described as follows: A set of $n$ rankable objects appear before us one at a time in random order with $n!$ permutations equally likely. As each objects appears, we must decide to select or reject it based on the relative ranks of the objects. The payoff is the length of time we are in possession of a relatively best object. Thus we will only select a relatively best object, receiving a payoff of one as we do so and an additional one for each new observation as long as the selected object remains relatively best.

Though Ferguson, Hardwick and Tamaki [2] considered the various duration models extensively, they confined themselves to the study of the one choice duration problem. In this paper, we attempt to extend the one choice problems to the multiple choice problems. For the $m$ choice duration problem, we are allowed to choose at most $m$ objects sequentially, and receive each time a unit payoff as long as either of the chosen objects remains a candidate (for simplicity we refer to a relatively best object as a candidate). Obviously


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only candidates can be chosen, the objective being to maximize expected payoff. This problem can be viewed from another perspective as follows: Let $T(i)$ be defined as the time of first candidate after time $i$ if there is one, and $n+1$ if there is none. Then $T(i)  - i$ is duration of the candidate selected at time $i$ and the objective is to find a stopping vector $(\tau_1^*, \tau_2^*, \ldots, \tau_m^*)$ such that
\[
E \left[ \sum_{i=1}^{m} (T(\tau_i^*) - \tau_i^*) \right] = \sup_{(\tau_1, \tau_2, \ldots, \tau_m) \in C_m} E \left[ \sum_{i=1}^{m} (T(\tau_i) - \tau_i) \right],
\]
where $\tau_i^*, 1 \leq i \leq m$, denotes the stopping time related to the $i$th choice and $C_m$ is the set of all possible vectors $(\tau_1, \tau_2, \ldots, \tau_m)$.

This problem will be discussed in Section 2. We formulate this as a Markovian decision process model and derive the optimal strategy in Section 2.1. It can be shown that, for the $m$ choice duration problem, there exists a sequence of integer-valued critical numbers $(s_1, s_2, \ldots, s_m)$ such that, whenever there remain $k$ choices yet to be made, then the optimal strategy immediately selects a candidate if it appears after or on time $s_k$, $1 \leq k \leq m$. It is also shown that $s_k$ is non-increasing in $k$. In Section 2.2 we investigate the asymptotics as $n \to \infty$. $s_k/n$ proves to converge to some definite value $s_k^*$ and a recursive formula for calculating $s_k^*$ in terms of $s_1, s_2, \ldots, s_{k-1}$ will be given by
\[
s_k^* = \exp \left\{ - \left( 1 + \sqrt{1 - 2 \sum_{i=1}^{k-1} \frac{(k-i+1)+ (k-i+1) \log s_i^*}{(k-i+2)!} (\log s_i^*)^{k-i+1}} \right) \right\}.
\] (1)
The vacuous sum is assumed to be zero throughout the paper. Thus the above formula is valid for $k \geq 1$. It is also shown that, as $n \to \infty$, (the maximum expected payoff)/$n$ converges to $-\sum_{k=1}^{m} s_k^* \log s_k^*$.

The best-choice secretary problem is concerned with maximizing the probability of choosing the best object from among all. In Section 2.3, we show that the multiple choice duration problem with a known number of $n$ objects is equivalent to the multiple choice best-choice secretary problem with unknown number of objects having a uniform distribution on $\{1, 2, \ldots, n\}$. Pressman and Sonin [9] considered the one choice problem with unknown number of objects. Tamaki [12] formulated the multiple choice problem with an unknown number of objects and solved the two choice problem having a uniform prior on the actual number of objects. See also Petracci [7] and Lehtinen [4] for the problem with an unknown number of objects and Gilbert and Mosteller [3], Sakaguchi [10] and Preater [8] for the multiple choice problem.

In Sections 3 and 4, the multiple choice duration problem treated in Section 2 is generalized by introducing cost. In Section 3, we consider a problem in which a constant acquisition cost is incurred each time an object is chosen. Thus far we have assumed that the objects, once chosen, are possessed until the process terminates. Instead we consider in Section 4 a problem which allows us to possess only one object at a time. A constant replacement cost is incurred each time we replace a previously chosen object with a new one. The objectives are, in Sections 3 and 4, to maximize the expected net payoff. It can be shown that, under an appropriate cost condition, the optimal strategies have the same structure as that for the problem involving no cost.

The models we consider here are so called no information model where the decision to select an object is based only on the relative ranks of the objects observed so far. Another typical model left for the future research is full information model where the observations are the actual values of the objects assumed to be independent and identically distributed from a known distribution and hence the decisions is based on the actual values of the objects.

For a history and review of the secretary problem, the reader is referred to Ferguson [1] and Samuels [11].
2 Multiple choice duration problem

We assume that all that can be observed are the relative ranks of the objects as they are presented. Thus if \( X_i \) denotes the relative rank of the \( i \)-th object among those observed so far (\( i \)-th object is a candidate if \( X_i = 1 \)), the sequentially observed random variables are \( X_1, X_2, \ldots, X_n \). It is well known, under the assumption that the objects are put in random order with all \( n! \) permutations equally likely, that

(a) the \( X_i \) are independent random variables and

(b) \( P(X_i = j) = 1/i \), for \( 1 \leq j \leq i, 1 \leq i \leq n \).

2.1 The finite horizon problem

We consider the \( m \) choice duration problem as a Markovian decisions process model. Since serious decisions of either selection or rejection takes place only when a candidate appears, we describe the state of the process as \( (i, k) \), \( 1 \leq i \leq n, 1 \leq k \leq m \) if the \( i \)-th object is a candidate and there remain \( k \) more choices to be made. For the above process to be a Markov chain, we must further introduce an additional absorbing state \( (n + 1, k) \) denoting the situation where the process comes to an end with \( k \) choices left, \( 1 \leq k \leq m \).

The transition probability of this Markov chain is specified by the probability law of \( T(i) \). Let \( p(i, j) = P(T(i) = j) \). Then it is easy to see that from (a) and (b)

\[
p(i, j) = \begin{cases} \frac{i}{(i-1)j}, & j = i + 1, \ldots, n \\ \frac{1}{n}, & j = n + 1 \end{cases}
\]

and that, after leaving state \( (i, k) \) the process makes transition into state \( (j, k - 1) \) or \( (j, k) \), \( i < j \leq n + 1 \) with probability \( p(i, j) \) depending on whether the \( i \)-th object is selected or rejected.

The expected duration of the candidate selected in \( (i, k) \) is given by \( E[T(i) - i] \), which is calculated from (ref):2 as

\[
E[T(i) - i] = \sum_{j=i+1}^{n+1} (j-i)p(i, j) = \sum_{j=i}^{n} \frac{i}{j}.
\]

To make the solutions of the models more easily comparable to each other, we hereafter choose to maximize the expected total proportional durations of the chosen objects rather than the expected total durations. This implies that the expected contribution of the candidate selected in \( (i, k) \) is evaluated as \( E[T(i) - i]/n \) instead of \( E[T(i) - i] \).

Let \( W_i^{(k)} \) be the expected additional payoff under an optimal strategy starting from state \( (i, k) \), \( 1 \leq i \leq n, 1 \leq k \leq m \), and also let \( U_i^{(k)}(V_i^{(k)}) \) be the expected additional payoff when the expected additional payoff when we select (reject) the \( i \)-th object and then continues search in an optimal manner. Then the principle of optimality yields, for \( 1 \leq k \leq m \)

\[
W_i^{(k)} = \max\{U_i^{(k)}, V_i^{(k)}\}, \quad 1 \leq i \leq n,
\]

where

\[
U_i^{(k)} = E\left[ \frac{T(i) - i}{n} + W_{T(i)}^{(k-1)} \right] = \frac{1}{n} \sum_{j=i}^{n} \frac{i}{j} + \sum_{j=i+1}^{n} \frac{i}{j(j-1)} W_j^{(k-1)}
\]

and

\[
V_i^{(k)} = E[W_{T(i)}^{(k)}] = \sum_{j=i+1}^{n} \frac{i}{j(j-1)} W_j^{(k)}.
\]
Equations (3)-(5), combined with the boundary condition \( W_i^{(0)} = 0 \), \( 1 \leq i \leq n \), can be solved recursively to yield the optimal strategy and the optimal value \( W_i^{(m)} \). When \( n \leq m \), the optimal strategy never fails to achieve unit payoff by selecting the candidates successively as they appear. Thus we assume \( n > m \) unless otherwise specified. The optimal strategy can be summarized as follows.

**Theorem 2.1.** For the \( m \) choice duration problem, there exists a sequence of integer-valued critical numbers \( (s_1, s_2, \ldots, s_m) \) such that, whenever there remain \( k \) choices yet to be made, i.e., we have already chosen \( m - k \) objects, then the optimal strategy immediately selects a candidate if it appears after or on time \( s_k \).

Moreover, \( s_k \) is non-increasing in \( k \) and determined by

\[
s_k = \min\{i : G_i^{(k)} \geq 0\},
\]

where \( G_i^{(k)} \), \( 1 \leq i \leq n \), \( 1 \leq k \leq m \) is defined recursively as

\[
G_i^{(k)} = G_i^{(1)} + \sum_{j=\max(i+1,s_{k-1})}^{n} \frac{1}{j-1} G_j^{(k-1)}, \quad k \geq 2
\]

starting with

\[
G_i^{(1)} = \sum_{j=1}^{n} \frac{1}{j} - \sum_{j=i+1}^{n} \frac{1}{j-1} \sum_{l=1}^{n} \frac{1}{l}.
\]

**Proof.** Define \( G_i^{(k)} \), \( k \geq 1 \), as

\[
G_i^{(k)} = G_i^{(1)} + \sum_{j=i+1}^{n} \frac{n}{j(j-1)} \{W_j^{(k-1)} - V_j^{(k-1)}\}, \quad k \geq 2
\]

starting with \( G_i^{(1)} \) given in (8). We will naturally find in the course of the proof that this definition in fact agrees with that given in (7). Suppose that we are in state \((i, k)\). If we select a current candidate we receive \( U_i^{(k)} \). If instead, we continue and select the next candidate if any, we expect to receive \( \tilde{V}_i^{(k)} \) defined as

\[
\tilde{V}_i^{(k)} = E[U_i^{(k)}] = \sum_{j=i+1}^{n} \frac{i}{j(j-1)} U_j^{(k)} = \sum_{j=i+1}^{n} \frac{i}{j(j-1)} \left( \frac{1}{n} \sum_{l=j}^{n} \frac{1}{l} + V_j^{(k-1)} \right).
\]

The one-stage look ahead rule immediately calls for selection if \( U_i^{(k)} \geq \tilde{V}_i^{(k)} \). Then the strategy specified by (6), (9) and (8) is in fact the one-stage look ahead rule, because \( G_i^{(k)} \) can be written, from (4) and (10), as

\[
G_i^{(k)} = \left( \frac{n}{i} \right) \{U_i^{(k)} - \tilde{V}_i^{(k)}\}.
\]

Since the horizon is finite, the one-stage look-ahead rule is optimal if the problem is monotone. To prove that the problem is monotone and that \( s_k \) is non-increasing in \( k \), it suffices to show that, for each \( k \), \( G_i^{(k)} \) has the following properties:

(P1)_k \quad If \( G_i^{(k)} \geq 0 \), then \( G_{i+1}^{(k)} \leq 0 \) for \( 1 \leq i \leq n \)

(P2)_k \quad \( G_i^{(k+1)} \geq G_i^{(k)} \) for \( 1 \leq i \leq n \)

(P1)_k implies that the \( k \) choice problem is monotone and (P2)_k, combined with definition (6), guarantees
\( s_{k+1} \leq s_k \). We show \((P1)_k\) and \((P2)_k\) simultaneously by induction on \(k\). \((P1)_1\) holds since \(G^{(1)}_i = 1/n > 0\). \((P2)_1\) is immediate since, from (9),

\[
G^{(2)}_i = G^{(1)}_i + \sum_{j=j+1}^{n} \frac{n}{j(j-1)} \{W^{(1)}_j - V^{(1)}_j\} \geq G^{(1)}_i.
\]

Assume now that \(P(1)_j\) and \(P(2)_j\) hold for \(j = 1, 2, \ldots, k\). Then, considering that the one-stage look-ahead rule of the \(k\) choice problem is optimal from the induction hypothesis \((P1)_k\), we have that, for \(s_k = \min\{i : G^{(k)}_i \geq 0\}\) as defined in (6),

\[
W^{(k)}_j = \begin{cases} 
V^{(k)}_j, & j < s_k \\
U^{(k)}_j, & j \geq s_k
\end{cases}
\]

and

\[
V^{(k)}_j = \bar{V}^{(k)}_j, \quad j \geq s_k - 1.
\]

These are combined to yield, from (11),

\[
W^{(k)}_j - V^{(k)}_j = \begin{cases} 
0, & j < s_k \\
\frac{j}{n} G^{(k)}_j, & j \geq s_k
\end{cases}
\]

and applying this to (9) (with \(k\) replaced by \(k + 1\)) gives

\[
G^{(k+1)}_i = G^{(1)}_i + \sum_{j=\max(i+1, s_k)}^{n} \frac{1}{j-1} G^{(k)}_j.
\]  

(12)

It easy to see \(G^{(k)}_i \geq 0\) for \(i \geq s_k\) from the induction hypothesis \((P1)_k\) and \((P2)_k\). Thus, to prove \((P1)_{k+1}\), it suffices to show that \(G^{(k+1)}_i\) is increasing in \(i\) for \(i \leq s_k\). This is immediate from (12) since \(G^{(1)}_i\) is increasing in \(i\) for \(i \leq s_1\) and \(s_k \leq s_1\) through the induction hypotheses \((P2)_j\) for \(j = 1, 2, \ldots, k - 1\).

Now we turn to \((P2)_{k+1}\). Since the one-stage look-ahead rule of the \(k + 1\) choice problem is optimal from \((P1)_{k+1}\), we come to have, for \(s_{k+1} = \min\{i : G^{(k+1)}_i \geq 0\}\),

\[
G^{(k+2)}_i = G^{(1)}_i + \sum_{j=\max(i+1, s_{k+1})}^{n} \frac{1}{j-1} G^{(k+1)}_j,
\]  

(13)

in a similar way as (12) was derived. Therefore from (12) and (13)

\[
G^{(k+2)}_i - G^{(k+1)}_i = \sum_{j=\max(i+1, s_{k+1})}^{n} \frac{1}{j-1} G^{(k+1)}_j - \sum_{j=\max(i+1, s_k)}^{n} \frac{1}{j-1} G^{(k)}_j \geq
\]

\[
\sum_{j=\max(i+1, s_k)}^{n} \frac{1}{j-1} \{ G^{(k+1)}_j - G^{(k)}_j \} \geq \{use \ s_{k+1} \leq s_k \ from \ the \ induction \ hypothesis \ (P2)_k \}
\]

\[
\geq 0, \quad \{again \ from \ the \ induction \ hypothesis \ (P2)_k\}.
\]

Which proves \((P2)_{k+1}\) and hence completes the induction.
Let $q_m, m \geq 1$, be the expected payoff for the $m$ choice problem, i.e., $q_m \equiv W_1^{(m)}$. Then, from the property of the optimal strategy, we have
\[
q_m = V_{s_{m-1}}^{(m-1)} = \frac{s_{m-1} - 1}{n} \sum_{j=1}^{n} \frac{1}{j} \sum_{l=1}^{n} \frac{1}{l} + \frac{s_{m-1} - 1}{j(j-1)} V_j^{(m-1)},
\]
where $V_i^{(m-1)}, m \geq 2$, are calculated recursively as
\[
V_i^{(m-1)} = \begin{cases} 
q_{m-1}, & i < s_{m-1} - 1 \\
\frac{1}{n} \sum_{j=i+1}^{n} \frac{1}{j} \sum_{l=1}^{n} \frac{1}{l} + \frac{1}{j(j-1)} V_j^{(m-2)}, & i \geq s_{m-1} - 1 
\end{cases}
\]
with the interpretation that $V_i^{(0)} \equiv 0$.

2.2 Asymptotic results

It is of interest to investigate the asymptotic behaviors of $s_k, 1 \leq k \leq m$ and $q_m$ as $n$ tends to infinity. To do this, we here employ an intuitive approach of approximating the infinite sum by the corresponding integral. When $m = 1$, we easily find that, if one lets $i/n \rightarrow x$ as $n \rightarrow \infty$ in (8), $G_i^{(1)}$ is a Riemann approximation to the integral
\[
G_i^{(1)}(x) = \int_x^1 \frac{dy}{y} - \int_x^1 \frac{dy}{y} \int_y^1 \frac{dz}{z} = \frac{2 + \log x}{2}. 
\]
Thus, from (6), $s_{1}^{*} = \lim_{n \rightarrow \infty} \frac{2}{n} = e^{-2}$ is obtained as a unique root $x \in (0, 1)$ of the equation $G_i^{(1)}(x) = 0$.

Define in general $s_k^{*} = \lim_{n \rightarrow \infty} \frac{2}{n}$. Then, in similar way, we can obtain $s_k^{*}$ for $k \geq 2$ successively as a unique root $x \in (0, s_{k-1}^{*})$ of the equation
\[
G_i^{(k)}(x) = 0
\]
if $G_i^{(k)}(x), 0 < x < 1$, are defined recursively as
\[
G_i^{(k)}(x) = G_i^{(1)}(x) + \int_{\max(x, s_{k-1}^{*})}^1 \frac{1}{y} G_i^{(k-1)}(y) dy, \quad k \geq 2
\]
starting with $G_i^{(1)}(x)$ (note that $G_i^{(1)}$ is a Riemann approximation to $G_i^{(k)}(x)$ if one lets $i/n \rightarrow n$ as $n \rightarrow \infty$).

From (17) and (18), $s_k^{*}$ is a root of the equation
\[
G_i^{(1)}(x) = -\int_{s_{k-1}^{*}}^1 \frac{1}{y} G_i^{(k-1)}(y) dy,
\]
or equivalently, from (16)
\[
s_k^{*} = \exp \left\{ -\left( 1 + \sqrt{1 + 2 \int_{s_{k-1}^{*}}^1 \frac{G_i^{(k-1)}(y)}{y} dy} \right) \right\}.
\]

To derive the tractable form as given in (1), we need some lemmas.
Lemma 2.1. Define, for a positive integer \( k \geq 1 \),
\[
A_{k,i} = \int_{s_{k-i}^{*}}^{1} \frac{(\log x)^{i}}{x} G^{(k-i)}(x) dx, \quad 0 \leq i \leq k - 1
\]
\[
a_{k,i} = \int_{s_{k-i}^{*}}^{1} \frac{(\log x)^{i}}{x} G^{(1)}(x) dx, \quad 0 \leq i \leq k - 1.
\]
(20)

Then \( A_{k,i} \) satisfies the following recursive relation
\[
A_{k,i} = a_{k,i} + \left( \frac{1}{i+1} \right) [A_{k,i+1} - (\log s_{k-i}^{*})^{i+1} A_{k-i-1,0}],
\]
with the interpretation that \( A_{k,k} = 0, k \geq 0 \).
(21)

Proof. From (18)
\[
G^{(k-i)}(x) = \begin{cases} 
G^{(1)}(x) + A_{k-i-1,0}, & x < s_{k-i}^{*} \\
G^{(1)}(x) + \int_{s_{k-i}^{*}}^{1} \frac{1}{y} G^{(k-i-1)}(y) dy, & x \geq s_{k-i}^{*}.
\end{cases}
\]
Thus, since \( s_{k-i}^{*} \leq s_{k-i-1}^{*} \)
\[
A_{k,i} = \int_{s_{k-i}^{*}}^{s_{k-i-1}^{*}} \frac{(\log x)^{i}}{x} [G^{(1)}(x) + A_{k-i-1,0}] dx + \int_{s_{k-i}^{*}}^{1} \frac{(\log x)^{i}}{x} \left[ G^{(1)}(x) + \int_{s_{k-i}^{*}}^{1} \frac{1}{y} G^{(k-i-1)}(y) dy \right] dx =
\]
\[
= a_{k,i} + A_{k-i-1,0} \int_{s_{k-i}^{*}}^{s_{k-i-1}^{*}} \frac{1}{x} (\log x)^{i} dx + \int_{s_{k-i}^{*}}^{1} \left[ \int_{s_{k-i}^{*}}^{y} \frac{(\log x)^{i}}{x} dx \right] \frac{1}{y} G^{(k-i-1)}(y) dy.
\]
(22)

Since the second and third terms of (22) are respectively calculated as
\[
\left( \frac{A_{k-i-1,0}}{i+1} \right) [(\log s_{k-i-1}^{*})^{i+1} - (\log s_{k-i}^{*})^{i+1}]
\]
and
\[
\left( \frac{1}{i+1} \right) \int_{s_{k-i}^{*}}^{1} [(\log y)^{i+1} - (\log s_{k-i-1}^{*})^{i+1}] \frac{1}{y} G^{(k-i-1)}(y) dy
\]
\[
= \left( \frac{1}{i+1} \right) [A_{k,i+1} - A_{k-i-1,0} (\log s_{k-i-1}^{*})^{i+1}],
\]
applying these to (22) yields the desired result.

For simplicity, let \( A_{k,0} \) be denoted by \( A_{k} \). Then the repeated use of (21) immediately gives the following recursive relation of \( A_{k} \).

Lemma 2.2. \( A_{k}, k \geq 1 \) satisfies the following recursive relation
\[
A_{k} = \sum_{i=1}^{k} \left[ \frac{a_{k,i}}{k-i} - \frac{(\log s_{k-i}^{*})^{k-i+1}}{(k-i+1)} A_{k-i-1} \right].
\]
Let \( N_{k}, k \geq 1 \) be defined as
\[
N_{k} = -(1 + \sqrt{1 + 2A_{k-1}}).
\]
(23)

Then from (19)
\[
s_{k}^{*} = \exp(N_{k})
\]
(24)

and we have the following lemma.
Lemma 2.3. \( N_k, k \geq 1 \) satisfies the following recursive relation

\[
N_k = \left[ 1 + \sqrt{1 - 2 \sum_{i=1}^{k-1} \frac{(k-i+2) + (k-i+1)N_i(N_i)^{k-i+1}}{(k-i+2)!}} \right].
\] (25)

Proof. Straightforward calculation from (20) yields

\[
a_{k-1, k-i-1} = \frac{(N_i)^{k-i-1}}{k-i+1} \cdot \frac{2(k-i+2)}{(N_i)^{k-i+2}},
\] (26)

and gives

\[
A_{k-1} = \frac{2N_i + N_i^2}{2}.
\] (27)

Hence, using (26) and (27), we have from Lemma 2.2

\[
A_{k-1} = \sum_{i=1}^{k-1} \frac{a_{k-1, k-i-1}}{(k-i+1)!} \cdot \frac{(N_i)^{k-i}}{(k-i)!} \cdot A_{i-1}
\]

\[
= - \sum_{i=1}^{k-1} \frac{(k-i+2) + (k-i+1)N_i(N_i)^{k-i+1}}{(k-i+2)!}
\]

which, combined with (23), completes the proof.

Recursive formula (1) is an immediate consequence from 25 through (24). From (1) we successively have

\[
s_1^* = \exp(-2) \approx 0.1353
\]

\[
s_2^* = \exp\left\{ -\left( 1 + \sqrt{\frac{7}{3}} \right) \right\} \approx 0.0799
\]

\[
s_3^* = \exp\left\{ -\left( 1 + \frac{1}{3} \sqrt{15 + 14 \sqrt{\frac{7}{3}}} \right) \right\} \approx 0.0493
\]

\[
s_4^* = \exp\left\{ -\left( 1 + \sqrt{\frac{31}{45} + \frac{2}{81} \left( 15 + 14 \sqrt{\frac{7}{3}} \right)^{3/2}} \right) \right\} \approx 0.0311.
\]

See Table 1 for \( s_1^* \) and \( s_4^* (\epsilon = 0) \).

Concerning the expected payoff, we have the following lemma.

Lemma 2.4. Let \( q_m^* = \lim_{n \to \infty} q_m \) for \( m \geq 1 \). Then

\[
q_m^* = -\sum_{k=1}^{m} s_k^* \log s_k^*.
\] (28)

Proof. For \( m = 1 \), we immediately have from (14)

\[
q_1^* = s_1^* \int_{s_1^*}^{1} \frac{dy}{y} \int_{y}^{1} \frac{dz}{z} = -s_1^* \log s_1^*.
\]
For \( m \geq 2 \), we have from (14) and (15)

\[
q_{m}^{*} = \frac{1}{s_{m}^{*}} \int_{s_{m}^{*}}^{1} \frac{dy}{y} \int_{z}^{1} \frac{dz}{z} + s_{m}^{*} \int_{s_{m}^{*}}^{1} \frac{1}{y^2} V^{(m-1)}(y)dy
\]

\[= \frac{s_{m}^{*}}{2} (\log s_{m}^{*})^2 + s_{m}^{*} \int_{s_{m}^{*}}^{1} \frac{1}{y^2} V^{(m-1)}(y)dy, \tag{29}\]

if \( V^{(m-1)}(x) \), \( 0 < x < 1, m \geq 2 \), are defined recursively as

\[
V^{(m-1)}(x) = \begin{cases} 
q_{m-1}^{*}, & 0 < x < s_{m-1}^{*} \\
x \int_{z}^{1} \frac{dy}{y} \int_{z}^{1} \frac{dz}{z} + x \int_{z}^{1} \frac{1}{y^2} V^{(m-2)}(y)dy, & s_{m-1}^{*} \leq x < 1
\end{cases}
\]

starting with \( V^{(0)}(x) \equiv 0 \) is a Riemann approximation to \( V^{(m-1)}(x) \) if one lets \( i/n \to x \) as \( n \to \infty \).

On the other hand, we have from (9)

\[
G_{x}^{(m)} = G_{x}^{(1)} + \left( \frac{n}{x} \right) \left[ V_{x}^{(m-1)} + \sum_{j=i+1}^{n} \frac{i}{j(j-1)} V_{x}^{(m-1)} \right],
\]

which yields, upon letting \( i/n \to x \) as \( n \to \infty \)

\[
G^{(m)}(x) = G^{(1)}(x) + \left( \frac{1}{x} \right) \left[ V^{(m-1)}(x) - \int_{x}^{1} \frac{1}{y^2} V^{(m-1)}(y)dy \right].
\]

Then \( G^{(m)}(s_{m}^{*}) = 0 \) implies that

\[
G^{(1)}(s_{m}^{*}) + \frac{1}{s_{m}^{*}} V^{(m-1)}(s_{m}^{*}) - \int_{s_{m}^{*}}^{1} \frac{1}{y^2} V^{(m-1)}(y)dy = 0,
\]

or equivalently, from (16) and \( V^{(m-1)}(s_{m}^{*}) = q_{m-1}^{*} \),

\[
\int_{s_{m}^{*}}^{1} \frac{1}{y^2} V^{(m-1)}(y)dy = -\frac{(2 + \log s_{m}^{*}) \log s_{m}^{*} + s_{m-1}^{*}}{2} \tag{30}
\]

Thus applying (30) to (29) yields

\[
q_{m}^{*} = -s_{m}^{*} \log s_{m}^{*} + q_{m-1}^{*}, \tag{28}
\]

which leads to (28) upon repetition.

Numerical values of the first four \( q_{m}^{*} \) are \( q_{1}^{*} = 0.2707, q_{2}^{*} = 0.4725, q_{3}^{*} = 0.6208, q_{4}^{*} = 0.7287 \). See Table 2 for \( q_{m}^{*} \) and \( q_{10}^{*}(c = 0) \).

Remark: The approach we took in the above is rather intuitive. To make the argument more rigorous,
we can appeal to the differential-equation approach developed by Mucci [5] and [6] (see also Yasuda [13]). We have, from (3)-(5),

\[ \frac{V_{i}^{(k)} - V_{i-1}^{(k)}}{1/n} = - \left( \frac{n}{i} \right) \sum_{j=1}^{i} \frac{1}{j} + V_{i}^{(k-1)} - V_{i}^{(k)} \]

Let \( i/n \to x \) as \( n \to \infty \). Then it is not difficult to show that, asymptotically, \( V_{i}^{(k)} \approx f^{(k)}(i/n) \), where

\[ \frac{d}{dx} f^{(k)}(x) = - \left( \frac{1}{x} \right) \left[ -x \log x + f^{(k-1)}(x) - f^{(k)}(x) \right]^{+}, \quad 0 \leq x \leq 1 \]

with boundary condition \( f^{(k)}(1) = 0 \), and a non-increasing sequence of critical numbers,

\[ s_{k}^{*} : f^{(k)}(s_{k}^{*}) = f^{(k-1)}(s_{k}^{*}) - s_{k}^{*} \log s_{k}^{*}, \quad k = 1, 2, \ldots \]

The function \( f^{(k)}(\cdot) \) is constant on \([0, s_{k}^{*}]\), so the expected payoff is \( q_{k}^{*} = f^{(k)}(0) = f^{(k)}(s_{k}^{*}) \). For example, some routine algebras yield, for \( k = 1 \) and \( 2 \),

\[ f^{(1)}(x) = \begin{cases} -s_{1}^{*} \log s_{1}^{*}, & 0 \leq x \leq s_{1}^{*} \\ \frac{x}{2} (\log x)^{2}, & s_{1}^{*} \leq x \leq 1 \end{cases} \]

\[ f^{(2)}(x) = \begin{cases} -s_{1}^{*} \log s_{1}^{*} - s_{2}^{*} \log s_{2}^{*}, & 0 \leq x \leq s_{2}^{*} \\ 2s_{1}^{*} - \frac{3}{2} x + \frac{3}{2} (\log x)^{2}, & s_{2}^{*} \leq x \leq s_{3}^{*} \\ \frac{x}{2} (\log x)^{2} - \frac{3}{2} (\log x)^{3}, & s_{3}^{*} \leq x \leq 1 \end{cases} \]

where \( s_{1}^{*} = \exp(-2) \) and \( s_{3}^{*} = \exp\{-1 + \sqrt{7}/3\} \). For \( k \geq 3 \), we can proceed in similar way.

### 2.3 Multiple choice secretary problem with random number of objects

Before concluding this section, we show that the multiple choice duration problem with a known number of \( n \) objects is equivalent to the multiple choice) best-choice secretary problem with an unknown number having a uniform distribution on \([1, 2, \ldots, n]\) in the sense that the optimal strategies and the expected payoffs are the same.

The general form of the \( m \) choice secretary problem with an unknown (bounded) number of objects is as follows: At most \( n \) objects appear before us, but we do not know exactly how many object will appear. We have only prior distribution \( p_{i} = P\{M = i\} \) on the actual number \( M \) of objects, where \( \sum p_{i} = 1 \).

We are allowed to make at most \( m \) choices and win if either one of the chosen objects is the best overall. The objective is to maximize the probability of win.

Tannaki[12] formulated this problem as a Markovian decision process model and explicitly solved the two choice problem with a uniform prior on \( M \). Let the state of the process be described as \((i, k)\), \( 1 \leq i \leq n \), \( 1 \leq k \leq m \), if the \( i \) th object is a candidate and there remain \( k \) choices yet to be made. Let also \( u_{i}^{(k)}(v_{i}^{(k)}) \) be the probability of win when we select (reject) the \( i \) th object and then continues optimally from state \((i, k)\). Then if we let

\[ w_{i}^{(k)} = \max \left\{ u_{i}^{(k)}, v_{i}^{(k)} \right\}, \quad 1 \leq i \leq n \]

\[ (31) \]
and put \( \pi_j = \sum_{i=1}^{j} p_i, \quad 1 \leq j \leq n, \) the principle of optimality yields

\[
\begin{align*}
\nu_i^{(k)} &= \sum_{i=j}^{n} \frac{i}{j} \pi_j + \sum_{j=i+1}^{n} \frac{i}{j(j-1)} \pi_j \nu_i^{(k-1)} \\
\nu_i^{(k)} &= \sum_{j=i+1}^{n} \frac{i}{j(j-1)} \pi_j \nu_j^{(k)}
\end{align*}
\]  

(32) (33)

Thus it is easy to see that, when \( M \) has uniform distribution on \( \{1, 2, \ldots, n\} \), i.e., \( p_i = \frac{1}{n}, \quad 1 \leq i \leq n, \) (31), (32) and (33) are transformed into (3), (4) and (5) respectively if we put \( U_i^{(k)} = \frac{n-i+1}{n} \nu_i^{(k)}, \quad V_i^{(k)} = \frac{n-i+1}{n} \nu_i^{(k)} \) and \( W_i^{(k)} = \frac{n-i+1}{n} \nu_i^{(k)} \).

3 Multiple choice duration problem with an acquisition cost

In this section, the multiple choice duration problem is generalized by imposing a constant acquisition cost \( c (\geq 0) \) each time an object is chosen. The objective of this problem is to maximize the expected net payoff, where the net payoff is defined as the payoff earned less the total acquisition cost incurred.

3.1 The finite horizon problem

We treat the \( n \) choice duration problem with an acquisition cost \( c \). Let the state of the process be defined as in Section 2, and let also \( W_i^{(k)}, \quad U_i^{(k)} \) and \( V_i^{(k)} \) be defined similarly as the expected additional net payoff under an optimal strategy starting from state \((i, k)\), \( 1 \leq i \leq n, 1 \leq k \leq m \). Then the optimality equations (3)-(5) still hold if (4) is replaced by

\[
U_i^{(k)} = -c + \frac{1}{n} \sum_{j=i}^{n} \frac{i}{j} + \sum_{j=i+1}^{n} \frac{i}{j(j-1)} W_i^{(k-1)}, \quad 1 \leq k \leq m.
\]  

(34)

Consider the one choice problem. It is easy to see that

\[
U_i^{(1)} = -c + \frac{1}{n} \sum_{j=i}^{n} \frac{i}{j}
\]  

(35)

is unimodal with mode at the value

\[
K(n) = \min \left\{ i : \sum_{j=i+1}^{n} \frac{i}{j} \leq 1 \right\}.
\]  

(36)

Thus, if \( c \) is so large to make \( U_i^{(1)}(K(n)) < 0 \), we do not pay \( c \) for selecting a candidate no matter where it appears. That is, the optimal strategy makes no choice. This can be said on the multiple choice problem, and henceforth we consider only the case \( U_i^{(1)}(K(n)) \geq 0 \), namely,

\[
c \leq \frac{K(n)}{n} \sum_{j=K(n)}^{n} \frac{1}{j}.
\]  

(37)

Let

\[
b(n) = \max \left\{ i : U_i^{(1)} \geq 0 \right\}.
\]  

(38)
Then it goes without saying that the optimal strategy selects no object after time \(b(n)\), and hence our attention can be concentrated on the candidates that appear no later than \(b(n)\). A bit of consideration shows that this remains true for the multiple choice problem. The following theorem summarizes the optimal strategy for the \(m\) choice problem with acquisition cost \(c\).

**Theorem 3.1.** For the \(m\) choice duration problem with the cost condition \((37)\), there exists a sequence of integer-valued critical numbers \((s_1, s_2, \ldots, s_n)\) such that, whenever there remain \(k\) choices yet to be made, then the optimal strategy immediately selects a candidate if it appears after or on time \(s_k\), but no later than \(b(n)\). Moreover, \(s_k\) is non-increasing in \(k\) and determined by

\[
s_k = \min\{i \leq b(n) : G_i^{(k)} \geq 0\},
\]

where \(G_i^{(k)}\), \(1 \leq i \leq b(n)\), \(1 \leq k \leq m\), is defined recursively as

\[
G_i^{(k)} = G_i^{(1)} + \sum_{j=\max\{(i+1,s_{k-1})\}}^{b(n)} \frac{1}{j-1} G_j^{(k-1)}, \quad k \geq 2
\]

starting with

\[
G_i^{(1)} = \frac{1}{j} - \frac{\sum_{j=1}^{b(n)} 1}{j-1} \sum_{l=1}^{n} \frac{1}{l} \frac{e}{b(n)}.
\]

**Proof.** We omit the proof because it is similar to that of Theorem 2.1 (the theorem just states that the one-stage look-ahead rule is optimal under the assumption the choices must be made no later than \(b(n)\)).

### 3.2 Asymptotic results

Observe first that, since \(\lim_{n \to \infty} \frac{K(n)}{n} = c^{-1}\) from \((36)\), the cost condition \((37)\) is reduced, as \(n \to \infty\), to

\[
e \leq c^{-1}.
\]

If one lets \(i/n \to x\) as \(n \to \infty\) in \((35)\), \(U_i^{(1)}\) approaches

\[
U^{(1)}(x) = -c + x \int_x^1 \frac{dy}{y} = -c - x \log x.
\]

Let \(\beta = \lim_{n \to \infty} \frac{K(n)}{n}\). Then \(\beta\) is a unique root \(x \in [c^{-1}, 1]\) of the equation \(U^{(1)}(x) = 0\) under the cost condition \((42)\) and satisfies

\[-\beta \log \beta = c.
\]

Define \(s_k^* = \lim_{n \to \infty} \frac{s_k}{n}\), \(k \geq 1\). When one lets \(i/n \to x\) as \(n \to \infty\) in \((41)\), \(G^{(1)}(x)\) approaches the integral

\[
G^{(k)}(x) = \int_x^1 \frac{dy}{y} - \int_x^{\beta} \frac{dy}{y} \int_y^1 \frac{dz}{z} - \frac{c}{\beta} = \frac{(2 + \log x) \log x - (2 + \log \beta) \log \beta}{2}.
\]

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Thus \( s^*_k = \exp\{-(2 + \log \beta)\} \) is obtained as a unique root \( x \in (0, \beta) \) of the equation \( G^{(1)}(x) = 0 \). Let \( G^{(k)}(x), 0 < x < \beta, 1 \leq k, \) be recursively defined as

\[
G^{(k)}(x) = G^{(1)}(x) + \int_{\max(x, s^*_{k-1})}^{\beta} \frac{1}{y} G^{(k-1)}(y) \, dy, \quad k \geq 2
\]

starting with \( G^{(1)}(x) \). Then, given \( G^{(k-1)}(x) \) and \( s^*_{k-1} \), \( s^*_k \) is obtained as a unique root \( x \in (0, s^*_{k-1}) \) of the equation \( G^{(k)}(x) = 0 \), or equivalently, from (44)

\[
s^*_k = \exp \left\{ - \left( 1 + \sqrt{(1 + \log \beta)^2 + 2 \int_{s^*_{k-1}}^{\beta} \frac{G^{(k-1)}(y)}{y} \, dy} \right) \right\}.
\]

We can show in a similar way as developed in the previous section that Lemma 2.2 still holds, that is we have

\[
A_{k-1} = \sum_{i=1}^{k-1} \frac{a_{k-1, k-i-1}}{(k-i-1)!} - \frac{(\log s^*_i)^{k-i} A_{i-1}}{(k-i)!},
\]

if one defines for a positive integer \( k \geq 1 \),

\[
A_k = \int_{s^*_1}^{\beta} \frac{G^{(k)}(x)}{x} \, dx,
\]

\[
a_{k, i} = \int_{s^*_i}^{\beta} \frac{(\log x)^i}{x} G^{(1)}(x) \, dx, \quad 0 \leq i \leq k - 1.
\]

Since straightforward calculation from (47) yields

\[
a_{k-1, k-i-1} = -\frac{(2 + \log \beta) \log \beta}{2(k-i)} B_{k, i} + \frac{1}{k-i+1} B_{k+1, i} + \frac{1}{2(k-i+2)} B_{k+2, i},
\]

where

\[
B_{k, i} = (\log s^*_i)^{k-i} - (1 + \log \beta)^{k-i}
\]

If \( k \) is replaced by \( i \), (45) can be written as

\[
A_{i-1} = \frac{1}{2} \left\{ (1 + \log s^*_i)^2 - (1 + \log \beta)^2 \right\}.
\]

Substituting these into (46) and then applying it to (45) gives the following recursive formula.

**Lemma 3.1.** When \( c \leq e^{-1}, s^*_k \) satisfies the following recursive relation

\[
s^*_k = \exp \left\{ - \left( 1 + \sqrt{(1 + \log \beta)^2 - 2 \sum_{i=1}^{k-1} \frac{[(k-i+2)B_{k+1, i} + (k-i+1)B_{k+2, i}]}{(k-i+2)!} } \right) \right\}.
\]

Let \( \gamma = 1 + \log \beta \). Then from Lemma 3.1 we can calculate \( s^*_k \) successively as follows.

\[
s^*_1 = \exp\{-1 + \gamma\}
\]

\[
s^*_2 = \exp\left\{ - \left( 1 + \gamma \sqrt{1 + \frac{2}{3} \gamma} \right) \right\}
\]

\[
s^*_3 = \exp\left\{ - \left[ 1 + \gamma \sqrt{1 + \frac{2}{3} \gamma \{1 + (1 + \frac{2}{3} \gamma)^{3/2}\}} \right] \right\}.
\]

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Let \( q^*_{m} \), \( m \geq 1 \), be the expected net payoff for the \( m \) choice duration problem when \( n \) tends to infinity. Then we have

**Lemma 3.2.** When \( c \leq e^{-1} \), we have for \( m \geq 1 \)

\[
q^*_{m} = -\left( \sum_{k=1}^{m} s^*_k \log s^*_k + mc \right).
\]

**Proof.** Omitted because the proof is similar to that of Lemma 2.4.

<table>
<thead>
<tr>
<th>Table 1: The asymptotic critical number ( s^*_m ) for some values of ( m ) and ( c ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c )</td>
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</tr>
<tr>
<td>0.0</td>
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<tr>
<td>0.1</td>
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<tr>
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<tr>
<td>0.3</td>
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</tbody>
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<table>
<thead>
<tr>
<th>Table 2: The asymptotic expected number net payoff for some values of ( m ) and ( c ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c )</td>
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</table>

Table 1 presents the numerical values of \( \beta \) and \( s^*_m \) for some values of \( m \) and \( c \). Let \( \beta' \) be the unique root \( x \in (0, e^{-1}) \) of the equation \(-x \log x = c\). Then it is intuitively clear that, as \( m \to \infty \), \( s^*_m \) converges to \( \beta' (= s^*_{m}) \) because it does no benefit to choose a candidate prior to \( \beta' \).

Table 2 presents the numerical values of \( q^*_m \) for some values of \( m \) and \( c \). It is interesting to compare, for example, \( q^*_1 = 0.0335 \) for \( c = 0.3 \) to \( q^*_1 = 0.2707 \) for \( c = 0 \), which implies that we can still gain positive payoff even when the acquisition cost is larger than the maximum payoff attainable when the acquisition cost is zero. This is not a contradiction. The stopping region shrinks as \( c \) gets large (see Table 1) and the positive payoff is yet assured by restricting our choice to a really good object. Table 2 also suggests that, as \( m \to \infty \), \( q^*_m \) converges to some value \( q^*_\infty \), which is given in the following lemma.

**Lemma 3.3.**

\[
q^*_\infty = (\beta - \beta') \left( 1 - \frac{c^2}{\beta^2 \bar{\gamma}} \right)
\]

**Proof.** We start with the following observation: As the arrival times of the \( n \) objects, we consider time points \( 1/n, 2/n, \ldots, n/n \) instead of \( 1, 2, \ldots, n \). Then when \( n \) tends to infinity the transition probability
\( p(x, y) = \frac{\lambda}{\lambda + 1} \) converges to the transition density \( p(x, y) = x/y^2 \) as \( i/n \to x \), \( j/n \to y \) (see (2)) and the candidates appear according to a non-homogeneous Poisson process with intensity function \( \lambda(x) = 1/x \) from (a), (b), in Section 2. That is, if we let \( N(a, b) \) denote the number of candidates that appear in time interval \( (a, b) \), then \( N(a, b) \) becomes a Poisson random variable with parameter \( \log(b/a) \) (see Theorem 1 of Gilbert and Mosteller [3]).

Let the \( T(x) \) denote the time of the first candidate after time \( x \) if there is one and 1 if there is none. It is easy to see from the above that \( T(x) \) has its density

\[
    f_{T(x)}(t) = p(x, t) = \frac{x}{t^2}
\]

(49)
on time interval \((x, 1)\) and its probability mass \( x \) on 1. Since, as \( m \) (number of choices) \(\to\) \( \infty \), the optimal strategy chooses all the candidates that appear in time interval \((\beta, \beta')\), the total proportional duration \( D \) is expressed as

\[
    D = \begin{cases} T(\beta) - T(\beta'), & \text{if } T(\beta') \leq \beta \\ 0, & \text{if } T(\beta') > \beta \end{cases}
\]

(50)

Hence considering that \( T(\beta) \) and \( T(\beta') \) are independent (this is not difficult to verify), we have by conditioning on \( T(\beta') \)

\[
    E[D] = E[T(\beta) - T(\beta') | T(\beta') \leq \beta] P(T(\beta') \leq \beta)
\]

\[
    = E[T(\beta)] P(T(\beta') \leq \beta) - E[T(\beta') | T(\beta') \leq \beta] P(T(\beta') \leq \beta)
\]

\[
    = \left( \int_\beta^1 t f_{T(\beta)}(t) dt + \beta \right) \left( \int_0^\beta f_{T(\beta')}(t) dt \right) - \left( \int_\beta^1 t f_{T(\beta)}(t) dt \right)
\]

\[
    = (c + \beta) \left( 1 - \frac{\beta'}{\beta} \right) - c \left( 1 - \frac{\beta'}{\beta} \right)
\]

\[
    = \beta - \beta'.
\]

Thus the expected net payoff \( q^*_n \) is calculated as

\[
    E[D - c N(\beta, \beta')] = (\beta - \beta') - c \log\left( \frac{\beta'}{\beta} \right)
\]

which yields (48).

4 Multiple choice duration problem with a replacement cost

Thus far we have implicitly assumed that the object, once chosen, are possessed until the process terminates. Instead, in this section, we are allowed to possessed only one object at a time and a constant cost \( d > 0 \) is incurred each time replacement takes place. Imagine a situation where we are possessing an object and a new candidate has just appeared. At this epoch we decide either to choose a new one, in which case we must release the previous object paying \( d \), or to reject it in which case continue possessing the previous one. For simplicity no acquisition cost is considered. The objective is to maximize the expected net payoff, where the net payoff is defined as the payoff earned less the total replacement cost incurred. The multiple choice duration problem with a replacement cost may be considered as a marriage and divorce problem with by interpreting the replacement cost as a compensation money.
4.1 The finite horizon problem

We treat the m choice duration problem with replacement cost d (the problem is here referred to as the m choice problem if we are allowed to make replacement of the objects up to m−1 times, m ≥ 2). Let the state of the process be defined as in Section 2, and let also \( W_i^{(k)} \), \( U_i^{(k)} \) and \( V_i^{(k)} \) be defined similarly as the expected additional net payoff under an optimal strategy starting from state \((i, k)\), \(1 ≤ i ≤ n\), \(1 ≤ k ≤ m\). Then the optimality equations (3)-(5) still hold if (4) is replaced by

\[
U_i^{(k)} = -d + \frac{1}{n} \sum_{j=i}^{n} \frac{i}{j(j-1)} W_j^{(k-1)}, \quad 1 ≤ k ≤ m-1
\]

\[
U_i^{(m)} = \frac{1}{n} \sum_{j=i}^{n} \frac{i}{j(j-1)} W_j^{(m-1)}.
\]

Observe that, once the first choice is made, our problem reduced to the \(m-1\) choice problem with an acquisition cost \(d\). Thus the main concern of this problem is to determine when to make the first choice.

If \(d > \frac{K(n)}{k} \sum_{j=k(n)}^{1} \frac{1}{j}\), where \(K(n)\) is as defined in (36), no replacement takes place and hence the m choice problem reduces to the one choice problem treated in Section 2.

In the case

\[
d ≤ \frac{K(n)}{n} \sum_{j=k(n)}^{n} \frac{1}{j},
\]

(51)

the optimal strategy can be summarized as follows

**Theorem 4.1.** For the m choice duration problem with the cost conditions (51), there exists a sequence of integer-valued critical numbers \((s_1, s_2, \ldots, s_{m-1}, t_m)\) such that the optimal strategy first selects a candidate that appears after or on time \(t_m\) and then it replaces the previously chosen object with a new candidate that appears after or on time \(s_k\), but no later than \(c(n)\) if \(m\) more replacements are available, \(1 ≤ k ≤ m-1\), where

\[
c(n) = \max \{i : U_i^{(1)} ≥ 0\}.
\]

(52)

Moreover, \(t_m ≤ s_{m-1}\) and \(s_k\) is non-increasing in \(k\) and these values are determined by

\[
t_m = \min \left\{ i ≥ c(n) : G_i^{(m)} ≥ \sum_{j=c(n)+1}^{n} \frac{1}{j-1} \sum_{l=j}^{n} \frac{1}{l} \frac{d}{c(n)} \right\}
\]

(53)

\[
s_k = \min \{i ≤ c(n) : C_i^{(m)} ≥ 0\}, \quad 1 ≤ k ≤ m-1,
\]

where \(C_i^{(k)}\), \(1 ≤ i ≤ c(n)\), \(1 ≤ k ≤ m\), is defined recursively as

\[
G_i^{(k)} = G_i^{(1)} + \sum_{j=\max(i+1, s_{i-1})}^{c(n)} \frac{1}{j-1} G_j^{(k-1)}, \quad k ≥ 2
\]

(54)

starting with

\[
G_i^{(1)} = \sum_{j=1}^{i} \frac{1}{j} - \sum_{j=i+1}^{n} \frac{1}{j-1} \sum_{l=j}^{n} \frac{1}{l} \frac{d}{c(n)}.
\]

(55)

**Proof.** The theorem just states that the one-stage look-ahead rule is optimal. The proof is omitted.
4.2 Asymptotic results

The cost condition (51) is reduced, as \( n \to \infty \), to

\[ d \leq e^{-1}. \]  \hspace{1cm} (56)

Let \( \delta = \lim_{n \to \infty} \frac{\Delta(n)}{n} \). Then, under the condition (56), \( \delta \) is a unique root \( x \in [e^{-1}, 1) \) of the equation

\[-x \log x = d. \]

We have the following result concerning the limiting values \( \mathbf{s}_k^* = \lim_{n \to \infty} \frac{\mathbf{s}_k}{n}, k \geq 1 \) and

\[ t_m^* = \lim_{n \to \infty} \frac{t_m}{n}. \]

**Lemma 4.1.** **labellem41** When \( d \leq e^{-1}, t_m^* \) is expressed in terms of \( s_k^* \) as

\[ t_m^* = \exp \left\{ - \left\{ 1 + \sqrt{(1 + \log s_m^*)^2 - (2 + \log \delta) \log \delta} \right\} \right. \]

where \( s_k^*, 1 \leq k \leq m, \) satisfies the following recursive relation

\[ s_k^* = \exp \left\{ - \left\{ 1 + \sqrt{(1 + \log \delta)^2 - 2 \sum_{i=1}^{k-1} \frac{(k - i + 2)B_{k+1,i} + (k - i + 1)B_{k+2,i}}{(k - i + 2)!} \right\} \right. \],  \hspace{1cm} (58)

where

\[ B_{k,i} = (\log s_i^*)^{k-i} - (\log \delta)^{k-i}. \]

**Proof.** (58) is evident from Lemma 3.1. (57) is also immediate from (4.1).

Let \( \lambda = 1 + \log \delta. \) Then from (57) and (58) we have

\[ t_1^* = \exp \left\{ - \left( 1 + \sqrt{1 + \frac{4}{3} \lambda^3} \right) \right\} \] \hspace{1cm} (59)

\[ t_2^* = \exp \left\{ - \left[ 1 + \sqrt{1 + \frac{2}{3} \lambda^3 \left( 1 + \left( 1 + \frac{4}{3} \lambda \right)^{3/2} \right) \right] \right\}. \]  \hspace{1cm} (60)

Let \( r_m^*, m \geq 2, \) be the expected net payoff for the \( m \) choice duration problem when \( n \) tends to infinity. Then we have

**Lemma 4.2.**

(i) **When** \( d > e^{-1}, r_m^* = 2e^{-2}. \)  \hspace{1cm} (62)

(ii) **When** \( d \leq e^{-1}, r_m^* = \left( \sum_{k=1}^{m-1} s_k^* \log s_k^* + t_m^* \log t_m^* + (m - 1)d \right). \)  \hspace{1cm} (63)

**Proof.** Omitted.

<table>
<thead>
<tr>
<th>( d )</th>
<th>( t_2^* )</th>
<th>( t_3^* )</th>
<th>( t_5^* )</th>
<th>( t_{10}^* )</th>
<th>( t_{\infty}^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0916</td>
<td>0.0656</td>
<td>0.0397</td>
<td>0.0270</td>
<td>0.0268</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1063</td>
<td>0.0885</td>
<td>0.0725</td>
<td>0.0684</td>
<td>0.0684</td>
</tr>
<tr>
<td>0.3</td>
<td>0.1243</td>
<td>0.1186</td>
<td>0.1154</td>
<td>0.1151</td>
<td>0.1151</td>
</tr>
</tbody>
</table>

Table 3: The asymptotic critical number \( t_m^* \) for some values of \( m \) and \( d \).
Table 4: The asymptotic expected net payoff $r^*_m$ for some values of $m$ and $d$.

<table>
<thead>
<tr>
<th>$d$</th>
<th>$r^*_2$</th>
<th>$r^*_4$</th>
<th>$r^*_5$</th>
<th>$r^*_10$</th>
<th>$r^*_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.4917</td>
<td>0.4934</td>
<td>0.5828</td>
<td>0.6166</td>
<td>0.6168</td>
</tr>
<tr>
<td>0.2</td>
<td>0.3435</td>
<td>0.3845</td>
<td>0.4146</td>
<td>0.4198</td>
<td>0.4198</td>
</tr>
<tr>
<td>0.3</td>
<td>0.2927</td>
<td>0.3017</td>
<td>0.3056</td>
<td>0.3059</td>
<td>0.3059</td>
</tr>
</tbody>
</table>

Tables 3 and 4 give the numerical values of $t^*_m$ and $r^*_m$ for some values of $m$ and $d$ respectively. The values of $s^*_m$ is given in Table 1 if $c$ is interpreted as $d$. Tables 3 and 4 suggest that, as $m \to \infty$, $t^*_m$ and $r^*_m$ converge to some values $t^*_\infty$ and $r^*_\infty$ respectively. The following lemma gives $t^*_\infty$ and $r^*_\infty$.

**Lemma 4.3.** Let $\delta'$ be the unique root $x \in [0, e^{-1}]$ of the equation $-x \log x = d$ for $d \leq e^{-1}$. Then

$$r^*_\infty = (\delta - \delta') \left( 1 - \frac{d^2}{\delta'} \right) - t^*_\infty \log t^*_\infty,$$

(64)

where

$$t^*_\infty = \exp \left\{ - \left[ 1 + \sqrt{1 - 2d \left( \frac{\delta - \delta'}{\delta'} \right) + (\delta - \delta')(\delta + \delta') \left( \frac{d}{\delta'} \right)^2} \right] \right\}$$

(65)

**Proof.** (65) is immediate from (57). (64) is also immediate from Lemmas 3.2, 3.3 and 4.2 (ii).

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**References**


