SINGLE-LEVEL STRATEGIES FOR FULL-INFORMATION BEST-CHOICE PROBLEMS, I

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ABSTRACT. Some best choice problems with full information and imperfect observation and their extension to two player competitive situation are discussed. Players observe one by one sequentially a sequence of \( \text{iid} \) random variables from a known continuous distribution with the objective of choosing one of the \( k \) largest. The observations of the random variables are imperfect and the player (or players) is (are) informed only of whether it is larger than or less than a previously determined decision level. The problem is to find the optimal decision level that maximizes the probability of achieving his objective. The solution of this one person game is derived. In the two player competitive situation two typical types of the optimal stopping games for choosing the best observation are formulated and transformed into continuous games on the domain \([0, \infty)^2\). Solutions of them are given. It is shown that our optimal stopping games are easy-to-state but hard-to-solve.

1. Introduction. The subject of the paper is a class of optimal stopping problems for a sequence of \( \text{iid} \) \( r.v.s \) with full-information (FI) but imperfect observation to guarantee the maximal probability of achieving some objects. In 1975 Enns [1] first considered and solved the problem where the objective is to choose the best \( r.v. \). Some generalization are posed and solved by Porosiński [8]. In Sakaguchi [11, 14] some more general problems are discussed. In Section 2 we formulate and solve the optimal stopping problem for finding the optimal threshold strategy that maximizes the probability of selecting one of the \( k \) largest \( r.v.s \). It is shown that the asymptotically optimal threshold strategy is to stop at the earliest \( r.v. \) that is larger than \( e^{-\frac{2k}{n}} \), where \( a_k \) is determined by the equation

\[
\sum_{j=k}^{\infty} \frac{a^j}{j(j+1)!} = \frac{1}{k}.
\]

In the subsequent two sections the optimal stopping problem for selecting the best \( r.v. \), i.e. the case \( k = 1 \), solved in Section 2, is extended to optimal stopping games where two players compete in selecting the best \( r.v. \). Two typical types of the optimal stopping games, where each player’s objective and information condition are, by the notation used in Sakaguchi [13],

1°): Selecting-best/ Players-priority/ Common/ Zero-sum, with FI

and

2°): Selecting-best/ Earlier-stop/ Each/ Non-zero-sum, with FI,

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respectively in Section 3 and 4.

In $1^0$) Players 1 and 2 observe a common iid sequence of r.v.s, which are sampled one by one sequentially. Facing each r.v., each player should accept the earliest r.v. that exceeds his decision level. If one player accepts a r.v. and the other rejects it, the game is left thereafter as the other player’s one-person game. If both players stop simultaneously at a r.v., then Player 1 accepts it, by his priority, and the game is left, thereafter, as Player 2’s one-person game. A single player who accepts the best r.v. is the winner, getting reward of one unit from the opponent. Player 1(2) wants to maximize (minimize) the expected payoff to Player 1. Related interesting results can be found in Neumann, Porosiński & Szajowski [7], Porosiński & Szajowski [9] and Majumdar [5]. The no-information case priority games have been considered by Enns & Ferenstein [2], Sakaguchi [12] and Szajowski [17].

In $2^0$) there are two independent iid sets of r.v.s. Players observe the private iid sequence of r.v.. Facing each r.v., each player should accept the earliest r.v. that exceeds his own decision level. A player who is the first to stop at the best one in his set of r.v.s is the winner. The case, where both players stop simultaneously at the best one in each set of r.v.s, is a draw. Players’ aims are to maximize his own winning probability. An interesting related work is Mazalov [6], where players’ objective are slightly different from the one in the present paper. Other related works, but in the null-information case, were done by Presman & Sonin [10], Fushimi [3] and Sakaguchi [12].

For both of cases $1^0$ and $2^0$, the game is transformed into a continuous game on the domain $(0, \infty)^2$. Numerical solutions to them are given.

The considered problems are related to the full-information best-choice problem considered by Enns [1], Sakaguchi [11, 15], Porosiński [8] and the game version of the problem considered by Neumann, Porosiński & Szajowski [7] and Sakaguchi & Saario [14].

2. **Best choice problems for choosing one of the $k$-bests.** Let $X_1, X_2, \ldots, X_n$ be a sequence of iid r.v.s obeying uniform distribution on the unit interval $[0, 1]$. The $X$’s are sequentially observed one by one, but the observation is imperfect and we can only know whether the observed r.v. is greater than or less than a prescribed level $z \in [0, 1]$. After $X_t$ is observed, we have to either accept or reject the observation. Our aim is to accept one of the $k$ bests among all r.v.s. Neither recall nor uncertainty of selection is allowed. In this paper we restrict ourselves to the strategies which reject the r.v.s less than $z$ and accept the earliest r.v. greater than $z$. We call a win the event in which we accept a r.v. satisfying the objective. The event in which either we fail to accept any r.v. or accept r.v. dissatisfying his objective is a loss. We are looking for the decision level $z$ which gives the maximum probability of win.

Let $P^{(k)}(z)$ be the probability of win for selecting one of the $k$ bests under the strategy with the decision level $z$. Then since the winning event by accepting an r.v. on or before the $(n - k)$-th is

$$
\bigcup_{m=1}^{k} \bigcup_{j=1}^{n-k} \{X_1, X_2, \ldots, X_{j-1} \leq z < X_j \cap \text{exactly } m-1 \text{ r.v. thereafter } > X_j\},
$$

we have

$$
(2.1) \quad P^{(k)}(z) = \sum_{m=1}^{k} P^{(k,m)}(z) + z^{n-k}(1 - z^k)
$$
where

\[ P^{(k,m)}(z) = \sum_{j=1}^{n-k} z^{j-1} \int_{0}^{1} \frac{(n-j)(1-x)^{m-1}x^{n-j-m+1}dx}{(m-1)} \]

\[ = z^n \sum_{r=k}^{n-1} z^{-(r+1)} \left( \frac{r}{m-1} \right) \int_{0}^{1} (1-x)^{m-1}x^{r-m+1}dx \]

is the probability that the accepted r.v. on or before the \((n-k)\)-th is the \(m\)-th best with \(1 \leq m \leq k\).

We use the following abbreviations in Theorem 1.

- p.w. is the probability of win with the decision level \(z\);
- a.d.l. (a.p.w.) is the asymptotic decision level (probability of win), when the decision level is such that \(z = e^{-a/n}\), with \(a > 0\) and \(n \to \infty\).

Let us define functions

\[ H_n(z) = \sum_{m=1}^{n} \frac{z^{-m} - 1}{m}, \]

\[ G_n(z) = \sum_{j=n}^{\infty} \frac{z^j}{j(j+1)!}. \]

**Theorem 1.** (i) For \(k = 1\) (i.e. selecting the best) the probability of win is \(P^{(1)}(z) = z^n H_n(z)\) and the optimal strategy is to choose \(z_0\), which is determined by the equation

\[ H_n(z) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{z^i}, \]

and the maximum of probability of win is \(n^{-1} \sum_{i=0}^{n-1} z_i\). Therefore

\[ a.d.l. = \exp\left( -\frac{a_1}{n} \right) \]

\[ a.p.w. = \frac{1 - e^{-a_1}}{a_1} \approx 0.5174 \]

where \(a_1 = 1.5029\) is a unique root in \((1, \infty)\) of the equation

\[ \int_{0}^{a} \frac{e^t - 1}{t} dt = \frac{e^a - 1}{a}, \]

i.e. \(G_1(a) = 1\).

(ii) For \(k = 2\) (i.e. selecting one of the two bests) the probability of win is

\[ P^{(2)}(z) = z^n[2H_n(z) - n(\frac{1}{z} - 1)], n \geq 2, \]

and the optimal strategy is to choose \(z_0\) which is determined by the equation

\[ H_n(z) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{z^i} + \frac{n - 1}{2} \frac{1}{z} - \frac{n}{2}. \]

The maximum of \(P^{(2)}(z)\) is

\[ P^{(2)}(z_0) = \frac{2}{n} \sum_{i=0}^{n-1} z_0^i - z_0^{n-1}. \]
Moreover we have

\begin{align*}
\text{a.d.l.} &= \exp\left(-\frac{a_2}{n}\right) \\
\text{a.p.w.} &= \frac{2(1 - e^{-a_2})}{a_2} - e^{-a_2} \approx 0.7265
\end{align*}

where \( a_2 \approx 2.0177 \) is a unique root in \((a_1, \infty)\) of the equation

\begin{equation}
\int_0^a \frac{e^t - 1}{t} \, dt = \frac{e^a - 1}{a} + \frac{a - 1}{2},
\end{equation}

i.e. \( 2G_2(a) = 1 \).

(iii) For \( k = 3 \) (i.e. selecting one of the three bests) the probability of win is

\begin{equation}
P^{(3)}(z) = z^n [3H_n(z) - \frac{n(n-1)}{2} \frac{z^{-2} - 1}{2} + \frac{n(n-5)}{2} \left(\frac{1}{z} - 1\right)], \quad n \geq 3,
\end{equation}

and the optimal strategy is to choose \( z_0 \) which is determined by the equation

\begin{equation}
H_n(z) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{i} + \frac{(n-1)(n-2)}{6} \frac{1}{z^2} - \frac{(n-1)(n-5)}{6} \frac{1}{z} - \frac{n+1}{3}.
\end{equation}

The maximum of \( P^{(3)}(z) \) is

\begin{equation}
P^{(3)}(z_0) = \frac{3}{n} \sum_{i=0}^{n-1} z_0^i + \frac{(n-1)(n-4)}{4} (z_0^{n-2} - z_0^n) + \frac{n-5}{2} z_0^{n-1}.
\end{equation}

Moreover we have

\begin{align*}
\text{a.d.l.} &= \exp\left(-\frac{a_3}{n}\right) \\
\text{a.p.w.} &= \frac{3(1 - e^{-a_3})}{a_3} - (2 + \frac{a_3}{2}) e^{-a_3} \approx 0.8355
\end{align*}

where \( a_3 \approx 2.4934 \) is a unique root in \((a_2, \infty)\) of the equation

\begin{equation}
\int_0^a \frac{e^t - 1}{t} \, dt = \frac{e^a - 1}{a} - \frac{2}{3} + \frac{a}{2} + \frac{a^2}{12},
\end{equation}

i.e. \( 3G_3(a) = 1 \).

Proof. From (2.1)-(2.2), proofs are made as follows.

In case (i) we have

\begin{equation}
P^{(1,1)}(z) = z^n \sum_{r=1}^{n-1} z^{-(r+1)} \int_{z}^{1} x^r \, dx = z^n \sum_{r=2}^{n} \frac{1}{m} \left(\frac{1}{z^m} - 1\right)
\end{equation}

and

\begin{equation}
P^{(1)}(z) = P^{(1,1)}(z) + z^{n-1}(1 - z) = z^n \left[ \sum_{r=2}^{n} \frac{1}{m} \left(\frac{1}{z^m} - 1\right) + \frac{1}{z} - 1 \right] = z^n H_n(z).
\end{equation}

This \( P^{(1)}(z) \) attains maximum at \( z = z_0 \) determined by the equation (2.5). Put \( z = e^{-\frac{a}{n}} \) and let \( n \to \infty \). Then we have

\begin{equation}
P^{(1)}(z) = z^n \sum_{m=1}^{n} \frac{1}{m} \left(\frac{1}{z^m} - 1\right) \to e^{-a} \int_0^a \frac{e^t - 1}{t} \, dt.
\end{equation}
The right-hand side of (2.18) is maximized by $a = a_1$ determined by the equation (2.7). This equation has a unique root $a_1$ in $(1, \infty)$ and so (2.6) follows.

In the proof of case (ii) we have

$$P^{(2,1)}(z) = z^n \sum_{r=2}^{n-1} z^{-(r+1)} \int_z^1 x^r dx = z^n \sum_{m=3}^{n} \frac{1}{m} \left( \frac{1}{z^m} - 1 \right),$$

$$P^{(2,2)}(z) = z^n \sum_{r=2}^{n-1} rz^{-(r+1)} \int_z^1 (1-x)x^{r-1} dx,$$

$$= z^n \left[ \sum_{r=2}^{n-1} \frac{1}{r+1} \left( \frac{1}{z^{r+1}} - 1 \right) - (n-2)(\frac{1}{z} - 1) \right].$$

Hence

$$P^{(2)}(z) = P^{(2,1)}(z) + P^{(2,2)}(z) + z^n(1 - z^2)$$

$$= z^n \left[ 2 \sum_{m=3}^{n} \frac{1}{m} \left( \frac{1}{z^m} - 1 \right) - (n-2)(\frac{1}{z} - 1) + \frac{1}{z^2} - 1 \right]$$

$$= z^n \left[ 2H_n(z) - n(\frac{1}{z} - 1) \right],$$

i.e. we get (2.8). Equations (2.9) and (2.10) are easy to prove.

Put $z = e^{-\frac{a}{n}}$ and let $n \to \infty$. We have

$$(2.19) \quad P^{(2)}(z) \to e^{-a} \left[ 2 \int_0^a \frac{e^t - 1}{t} dt - a \right].$$

Probability $P^{(2)}(z)$ is maximized at $a = a_2$ determined by the equation (2.12), because $g(a) = -\frac{1}{2} + \sum_{j \geq 2} \frac{a_j}{j(j+1)!} = -\frac{1}{2} + G_2(a)$ is strictly increasing, convex and $g(a_1) = -\frac{a_1 - 1}{2} < 0$.

To prove the case (iii) we have

$$P^{(3,1)}(z) = z^n \sum_{r=3}^{n-1} z^{-(r+1)} \int_z^1 x^r dx = z^n \sum_{m=4}^{n} \frac{z^{-m} - 1}{m},$$

$$P^{(3,2)}(z) = z^n \sum_{r=3}^{n-1} rz^{-(r+1)} \int_z^1 (1-x)x^{r-1} dx$$

$$= z^n \left[ \sum_{r=3}^{n-1} \frac{z^{-(r+1)} - 1}{r+1} - (n-3)(\frac{1}{z} - 1) \right],$$

$$P^{(3,3)}(z) = z^n \sum_{r=3}^{n-1} \frac{1}{z^{r+1}} \left( \frac{r}{2} \right) \int_z^1 (1-x)^2x^{r-2} dx$$

$$= z^n \sum_{r=3}^{n-1} \frac{1}{z^{r+1}} \left( \frac{r}{2} \right) \left\{ \frac{1 - z^{r-1}}{r-1} - 2\frac{1 - z^r}{r} + \frac{1 - z^{r+1}}{r+1} \right\}$$

$$= z^n \sum_{r=3}^{n-1} \frac{z^{-(r+1)} - 1}{r+1} - \frac{(n+2)(n-3)}{4} \frac{1}{z^2 + 1} + \frac{n(n-3)}{2} \frac{1}{z}.$$
Henceforth

\[ P^{(3)}(z) = \sum_{m=1}^{3} P^{(3,m)}(z) + z^{n-3}(1 - z^3) \]

\[ = z^n \left[ 3 \sum_{m=4}^{n} \frac{z^{-m} - 1}{m} - (n-3)(\frac{1}{z} - 1) - \frac{n+2(n-3)}{4}(\frac{1}{z^2} + 1) \right. \]

\[ + \left. \frac{n(n-3)}{2z} + \frac{1}{z^3} - 1 \right] \]

\[ = z^n \left[ 3H_n(z) - \frac{n(n-1)}{2} \frac{z^{-2} - 1}{z} + \frac{n(n-5)}{2} \frac{1}{z} - 1 \right]. \]

i.e. we get the formulae (2.13). The probability \( P^{(3)}(z) \) given by (2.13) is maximized at \( z_0 \) given by (2.14) and the maximum value of the probability is given by (2.15). Put \( z = e^{-\frac{a}{\theta}} \) and let \( n \to \infty \). We get from (2.13)

\[ (2.20) \quad P^{(3)}(z) \to e^{-a} \left[ 3 \int_{0}^{a} \frac{e^t - 1}{t} \, dt - 2a - \frac{a^2}{4} \right], \]

since

\[ n(n-1) \left\{ \frac{z^{-2} - 1}{2} - \frac{1}{z} + 1 \right\} = \frac{n(n-1)}{2} (e^\frac{a}{\theta} - 1)^2 \to \frac{a^2}{2}. \]

The limit (2.20) is maximized at \( a = a_3 \) which is determined by the equation (2.17). Let

\[ h(a) \equiv \int_{0}^{a} \frac{e^t - 1}{t} \, dt - \frac{e^a - 1}{a} + \frac{2}{3} - \frac{a}{2} - \frac{a^2}{12}. \]

We have \( h(a) = -\frac{1}{3} + G_3(a) \). This function is strictly increasing and convex with \( h(a_2) = \frac{1}{6} - \frac{a_2^2}{12} \approx -0.1726 < 0 \). Hence the equation (2.17) has a unique root in \((a_2, \infty)\). \( \square \)

It may not be difficult to solve the problem for general \( k \), but who can be so generous in selecting the 4-th best or worse r.v.s?

Due to its importance in real life the exact (i.e. non-asymptotic) numerical result for \( k = 1, 2 \) is given in Table 1 for small \( 3 \leq n \leq 100 \).

\[ \text{Remark 1. The strategy adopted in this paper is slightly different from one used in [1, 11]. However, the two kinds of strategies asymptotically give the same result.} \]

\[ \text{Remark 2. Around the problem for } k = 1, \text{ there is a famous and important result. If we consider more general strategies in which we determine a decreasing sequence} \]

\[ 1 > z_{1,n} > z_{2,n} > \ldots > z_{n-1,n} > z_{n,n} \equiv 0 \]

and stop at (i.e. accept) the earliest \( X_\sigma \) that satisfies \( X_\sigma > z_{a,n} \). It is shown that the optimal \( \{z_{i,n}^*\}_{i=1}^n \) is given by \( z_{i,n}^* = b_{n-i} \), where \( b_0 = 0, b_1 = 1/2 \) and \( b_j, j = 1, 2, \ldots, \), is a unique root of the equation \( H_j(b) = 1 \).

It should be noted that the optimal "downward barrier" is sequentially derived in each stage, depending only on how many r.v.s are left to be observed thereafter. The asymptotic
Table 1. Optimal decision level and probability of win for \( k = 1, 2 \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( k = 1 )</th>
<th>( k = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( z_0 )</td>
<td>( P^{(1)}(z_0) )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{2}{3} )</td>
</tr>
<tr>
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<td>.60632</td>
</tr>
<tr>
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</tr>
<tr>
<td>100</td>
<td>.98500</td>
<td>.51958</td>
</tr>
</tbody>
</table>

The probability of win, as \( n \to \infty \), was computed by Gilbert & Mosteller [4] as 0.580164 by high-speed computer and its explicit expression was found by Samuels [16] as

\[
P \left[ \bigcup_{i=1}^{n} \{ (X_j < b_{n-j}, j = 1, 2, \ldots, i - 1) \cap (X_i > b_{n-i}) \cap (X_i = \max_{1 \leq j \leq n} X_j) \} \right] \rightarrow e^{-c} + (e^c - c - 1) \int_{c}^{\infty} t^{-1} e^{-t} dt \approx 0.580166,
\]

where \( c \approx 0.80435 \) is determined by the equation

\[
\int_{0}^{c} t^{-1} (e^t - 1) dt = \sum_{j=1}^{\infty} \frac{c^j}{j \cdot j!} = 1.
\]

It is surprising that our optimal strategy in case (i) gives the a.p.w. \( \approx 0.5174 \) worse in a not large amount in spite of its very simple structure.

About the problem of finding the optimal "downward barrier" for \( k = 2 \) and 3 the numerical solution, by using dynamic programming, are available (see Sakaguchi & Saario [14]). As far as the authors know, no theoretical results has been obtained yet.

3. A zero-sum best choice game where players’ priority is given. A zero-sum game version of the discrete-time, full-information and imperfect observation best-choice problem is considered in this section. Two players 1 and 2 observe sequentially a sequence of \( n \) iid r.v.s from uniform distribution on \([0, 1]\), with each objective of choosing the largest r.v.. Let Player 1(2) chose a single level \( z(w) \), and he rejects r.v.s as long as they are less than \( z(w) \) and accepts the earliest r.v.s that is \( \geq z(w) \). If the earliest r.v. that is \( \geq z \lor w \) appears, Player 1 is given the priority to accept it and drops out from the game thereafter.

The player accepting the largest r.v. wins the game and he is paid a unit reward from the opponent. Player 1(2) wants to maximize (minimize) the player I’s expected payoff.
Let $M(z, w)$ stand for the payoff to Player 1 when the levels $z$ and $w$ are chosen. Then for $z < w$ we have

\[(3.1) M(z, w) = \sum_{s=1}^{n} z^{s-1} \int_{z}^{1} x^{n-s} dx - \sum_{s=1}^{n-1} \sum_{t=s+1}^{n} w^{t-s-1} \left\{ (w - z) \int_{w}^{1} y^{n-t} dy + \int_{w}^{1} dx \int_{z}^{1} y^{n-t} dy \right\}.\]

The first sum stands for the case where Player 1 is the first who accepts a r.v. in $[z, 1)$ and wins. The second (3rd) double sum stands for the case where Player 1 is the first who accepts a r.v. in $[z, w)$ (in $[w, 1)$ by his priority) and loses. The case where Player 2 is the first who accepts a r.v. cannot occur. The $x$ and $y$ represent the r.v.s accepted by Player 1 and 2, respectively. The expression (3.1) can be rewritten in the form

\[(3.1') \quad M(z, w) = z^{n} \sum_{i=1}^{n} \frac{z^{i-1}}{i} + w^{n} \sum_{j=1}^{n-j} \left\{ \sum_{s=1}^{n-j} \frac{(z/w)^{s}}{s!} \right\} \left\{ j^{-1}(w^{-j} - 1) - \frac{w - w^{-j-1}}{j} \right\} \].

Similarly for $z > w$ we have

\[(3.2) \quad M(z, w) = \sum_{s=1}^{n} w^{s-1} \int_{z}^{1} x^{n-s} dx + \sum_{s=1}^{n-1} \sum_{t=s+1}^{n} z^{t-s-1} (z - w) \int_{z}^{1} x^{n-t} dx - \sum_{s=1}^{n} w^{s-1} \int_{w}^{1} y^{n-s} dy - \sum_{s=1}^{n-1} \sum_{t=s+1}^{n} w^{t-s-1} \int_{z}^{1} dx \int_{z}^{1} y^{n-t} dy.\]

The first (4-th) sum is the case where Player 1 accepts a r.v. in $[z, 1)$ by his priority and wins (loses), and the second (3rd) one is the case where Player 2 is the first who accepts a r.v. in $[w, z)$ and loses (wins). The expression (3.2) can be rewritten in the form

\[(3.2') \quad M(z, w) = w^{n} \sum_{i=1}^{n-i-1} w^{-i} (1 - 2z^{i} + w^{i}) + z^{n} \left\{ \sum_{j=1}^{n-j} \frac{(z/w)^{j}}{j} \right\} \frac{z^{-j} - 1}{j} + w^{n} \sum_{j=1}^{n-j} (n - j) (z/w)^{j+1} \left\{ \frac{z^{-j} - 1}{j} - \frac{z^{-j-1} - 1}{j+1} \right\}.\]

Note that (3.1') and (3.2') coincide at $z = w$ and so the game is a continuous game on the unit square $0 \leq z, w \leq 1$.

Define the function for $a \in (0, \infty)$,

$$\Phi(a) \triangleq \int_{0}^{1} \frac{e^{at} - 1}{t} dt = \sum_{j=1}^{\infty} \frac{a^{j}}{j \cdot j!}.$$ 

This function is strictly increasing and convex function with $\Phi(0) = 0$, $\Phi'(a) = a^{-1}(e^{a} - 1)$ and $\Phi''(a) = a^{-2}\{1 - (1 - a)e^{a}\} = \sum_{k=2}^{\infty} \frac{a^{k-2}}{k(k-2)!} > 0$. 


Now let $z = e^{-a}$, $w = e^{-b}$, with $a, b > 0$ and $n \to \infty$. Then (3.1') - (3.2') becomes a continuous game on $[0, \infty)^2$, with the payoff function

$$M(a, b) = \begin{cases} 
\frac{b}{a-b}(e^{-b} - e^{-a}) + e^{-a}\Phi(a-b), & \text{if } 0 < b < a, \\
 \frac{b}{a-b}[(b-a)(\Phi(b) - \Phi(b-a)) - \Phi(b) - a\left(\frac{1}{2} - 1\right)]
+ e^{-a}\Phi(a) + 1 + \frac{a}{b} - 1, & \text{if } 0 < a < b,
\end{cases}$$

since the second sum in (3.1') tends to

$$-e^{-b}\int_0^1 du \left[ \int_0^{1-u} e^{-(b-a)v} dv \right] \frac{d}{du} \left( \frac{e^{bu} - 1}{u} \right)$$

$$= \frac{b}{a-b}(e^{-b} - e^{-a}) - e^{-a}\Phi(a - \Phi(a - b)),$$

and the first, second and third sums in (3.2') tend to $e^{-b}(\Phi(b) - 2\Phi(b-a))$,

$$(b-a)e^{-a}\int_0^1 \left[ \int_0^{1-u} e^{-(a-b)v} dv \right] \frac{e^{au} - 1}{u} du$$

$$= e^{-a}\left[ \Phi(a) - e^{-(b-a)}(\Phi(b) - \Phi(b-a)) \right]$$

and

$$e^{-b}\int_0^1 (1-u)e^{-(b-a)u} \frac{d}{du} \left\{ \frac{e^{au} - 1}{u} \right\} du$$

$$= e^{-b}\left[ a + (b-a-1)(\Phi(b) - \Phi(b-a)) - (b-a) \left( \frac{e^{b} - 1}{b} - \frac{e^{b-a} - 1}{b-a} \right) \right],$$

respectively.

Note that (3.3) is a continuous function on $[0, \infty)^2$ with value $ae^{-a}$ on the halfline $a = b > 0$ and we have

$$M(a, 0) = e^{-a}\Phi(a) = e^{-a}\int_0^a \frac{e^t - 1}{t} dt,$$

$$M(0, b) = -e^{-b}\Phi(b),$$

both of which are reasonable facts since setting $z$ or $w = 1$ means that the game is actually one-person game. Also (3.4) agrees with the result in Theorem 1(i). We get from (3.3)

$$M(a, b) \underset{(a \to \infty)}{\longrightarrow} 0, \ \forall b \in [0, \infty)$$

and

$$M(a, b) \underset{(b \to \infty)}{\longrightarrow} e^{-a}\Phi(a), \ \forall a \in [0, \infty),$$

where the latter is due to $be^{-b}(\Phi(b) - \Phi(b-a)) \underset{(b \to \infty)}{\longrightarrow} 1 - e^{-a}$. Setting $z = 0$ means that Player 1 should stop at the first observation and his a.p.w. is zero. Setting $w = 0$ means that Player 2 [Player 1] should stop at the first observation and drops out thereafter if it is in $(0, z) \cup (z, 1)$. 
We want to derive a saddle point and the saddle value for the payoff function \((3.3)\). By differentiating \((3.3)\) we get

\[
\frac{\partial}{\partial a} M(a,b) = \begin{cases} 
\frac{b}{(a-b)^2} [(a-b+1)e^{-a} - e^{-b}] \\
e^{-a} [\Phi'(a-b) - \Phi(a-b)], & 0 < b < a \\
e^{-a}(\Phi'(a) - \Phi(a)) \\
e^{-b}(\Phi'(b) - \Phi(b)) + e^{-b}\Phi(b-a), & 0 < a < b 
\end{cases}
\]

and

\[
\frac{\partial}{\partial b} M(a,b) = \begin{cases} 
\frac{b}{(a-b)^2} [(b-a+1)e^{-b} - e^{-a}], & 0 < b < a \\
e^{-b} [(b-a)(\Phi(b-a) - \Phi'(b-a)) - \Phi(b-a) + \Phi'(b)] \\
\frac{b}{a}(\Phi'(b) - 1 + b) + (\Phi(b) - \Phi(b-a)) + (\Phi(b) - \Phi'(b))], & 0 < a < b.
\end{cases}
\]

Computer result, on the basis of \((3.3)-(3.6)\), shows the following

**Theorem 2.** For the zero sum game on \([0,\infty)^2\) with the payoff function \((3.1)\), a saddle point \((a_0, b_0)\), with \(0 < a_0 < b_0\), exists and the saddle value is

\[
e^{-b_0} [(b_0 - a_0)(\Phi(b_0) - \Phi(b_0 - a_0) - \Phi(a_0) - a_0(b_0^{-1} - 1)] \\
e^{-a_0}(\Phi(a_0) + 1) + \frac{a_0}{b_0} - 1 \approx 0.3235,
\]

where \((a_0, b_0) \approx (1.57205, 2.99628)\) is determined by a simultaneous equation

\[
e^{-a}(\Phi(a) - \Phi'(a)) + e^{-b}(\Phi(b) - \Phi'(b)) = e^{-b}\Phi(b-a)
\]

\[
(\Phi(b) - \Phi(b-a)) + (\Phi(b) - \Phi'(b)) = (b-a)(\Phi(b) - \Phi'(b) - \Phi(b-a) + \Phi'(b-a)) \\
+ a^{-1}(e^b - 1 - b + b^2).
\]

The theorems shows that Player 1 sets his decision-level \(z_0 = e^{-\frac{a_0}{b}}\) higher than the opponent in the optimal play, and the saddle value is positive, reflecting Player 1's advantage over his opponent due to the higher priority in playing game.

4. A non-zero-sum best-choice game where winning requires earlier stop. A non-zero-sum game version of the discrete time, full information and imperfect observation best-choice problem is considered in this section. We first state the problem as follows:

1. There are two Players 1 and 2, and a sequence of \(n\) iid bivariate r.v.s \(\{(X_t, Y_t)\}_{t=1}^{n}\) from independent uniform distribution on \([0,1]^2\). Player 1(2) observes \(X_t(Y_t)\)'s sequentially one by one.

2. Let Player 1(2) choose a single level \(z(w)\). He rejects \(X_t(Y_t)\)'s as long as they are less than \(z(w)\) and he accepts the earliest r.v. that is \(X_\sigma \geq z \geq Y_\tau\), where \(\sigma\) and \(\tau\) are the stopping times of the players.

3. If one of the players stops (accepts), the other player is not informed of this fact and continues playing. We call a win for each player the event in which he gets to be the first to stop at the best one in his set of r.v.s. If the two players stop simultaneously at the best one in each player's set of r.v.s, both of them are the winners.

4. The aim of each player in the game is to determine his decision level by which the probability that he becomes a single winner is maximized.
Let \( M_1(z,w) \) \((M_2(z,w))\) stand for the probability of winning for Player 1(2) when the levels \( z \) and \( w \) are chosen. Then we have

\[
M_1(z,w) = \sum_{s=1}^{n} z^{s-1} \int_{z}^{1} x^{n-s} dx \left\{ w^s + \sum_{t=1}^{s-1} w^{t-1} \int_{w}^{1} (1-y^{n-t})dy \right\}.
\]

The first (second) term in \((4.1)\) is the case where Player 1(2) stops first and wins (loses). The \( x \) and \( y \) represent the r.v.s accepted by Player 1 and 2, respectively. The expression \((4.1)\) can be rewritten as

\[
(4.1') \quad M_1(z,w) = (zw)^n \sum_{j=1}^{n} \frac{z^{j-1}}{j} \left\{ w^{-n} - w^{-j} + w^{-j+1} - \sum_{i=j+1}^{n} \frac{w^{-i} - 1}{i} \right\}.
\]

\( M_2(z,w) \) is equal to \( M_1(z,w) \) with \( z \) and \( w \) interchanged.

By letting \( z = e^{-a/n}, w = e^{-b/n} \), with \( a, b > 0 \) and \( n \to \infty \), we obtain a non-zero-sum continuous game on \([0, \infty)^2\) with payoff functions

\[
M_1(a,b) = e^{-a} \left[ \Phi(a) - be^{-b} \int_{0}^{1} \Phi(at) \Phi'(bt)dt \right],
\]

\[
M_2(a,b) = e^{-b} \left[ \Phi(b) - ae^{-a} \int_{0}^{1} \Phi'(at) \Phi(bt)dt \right],
\]

because

\[
\int_{0}^{1} e^{at} - \frac{1}{t} dt \int_{t}^{1} e^{bu} - \frac{1}{u} du = \int_{0}^{1} e^{bu} - \frac{1}{u} du \int_{0}^{u} e^{at} - \frac{1}{t} dt = b \int_{0}^{1} \Phi(at) \Phi'(bt)dt.
\]

We note that \((4.2)\) gives

\[
M_1(a,0) = e^{-a} \Phi(a), \quad M_1(0,b) = 0
\]

\[
M_2(0,b) = e^{-b} \Phi(b), \quad M_2(a,0) = 0
\]

and

\[
M_1(a,a) = M_2(a,a) = e^{-a} \Phi(a) - \frac{1}{2} (e^{-a} \Phi(a))^2 = \frac{1}{2} \left[ 1 - (1 - e^{-a} \Phi(a))^2 \right],
\]
on the halfline \( a = b \geq 0 \). Again setting \( z = 1 \) \((w = 1)\) means that Player 1(2) does not play any role in the game. Also since

\[
e^{-a} \Phi(a)(1 - e^{-b} \Phi(b)) \leq M_1(a,b) \leq e^{-a} \Phi(a), \quad \forall b \in [0, \infty),
\]

we have, for all \( a \in [0, \infty), M_1(a,b) \to 0 \) and \( M_1(a,b) \to e^{-a} \Phi(a) \). That is, setting \( z = 0 \) \((w = 0)\) means that Player 1(2) stops at the first observation and a.p.w. is zero. Furthermore we find that

\[
M_1(a,b) + M_2(a,b) = 1 - (1 - e^{-a} \Phi(a))(1 - e^{-b} \Phi(b)),
\]

by an evident identity

\[
a \int_{0}^{1} \Phi'(at) \Phi(bt)dt + b \int_{0}^{1} \Phi'(bt) \Phi(at)dt = \Phi(a) \Phi(b).
\]
Now we find an equilibrium for a pair of payoff functions (4.2). First we find that

\[ e^a \frac{\partial M_1}{\partial a} = be^{-b} \int_0^1 (\Phi(at) - t\Phi'(at))\Phi'(bt)dt - (\Phi(a) - \Phi'(a)), \]

\[ \Phi(at) - t\Phi'(at) = \sum_{j=1}^{\infty} \frac{(at)^j}{j!(j+1)} \left( \frac{1}{j} - \frac{1}{a} \right), \quad \forall 0 \leq t \leq 1 \quad \text{and} \quad a > 0, \]

and

\[ \Phi(a) - \Phi'(a) = \sum_{j=1}^{\infty} \frac{a^j}{j(j+1)} - 1 = 0, \quad \iff \quad a = a_1 = 1.5029, \]

(see (2.7) in Theorem 1(i)). Note that the function \( \Phi(at) - t\Phi'(at) \) is not monotonic in \( t \) for any fixed \( a > a_1 \).

Let

\[ \psi(c) = e^a \frac{\partial M_1}{\partial a} \bigg|_{a=b=c} \]

\[ = e^{-c} \left( \frac{1}{2} (\Phi(c))^2 - \frac{1}{c} (\Phi(2c) - 2\Phi(c)) \right) - \Phi(c) + \Phi'(c), \]

because

\[ \int_0^1 t(\Phi'(at))^2 dt = \frac{1}{a^2} (\Phi(2a) - 2\Phi(a)) \]

and

\[ a \int_0^1 \Phi(at)\Phi'(at)dt = \frac{1}{2}(\Phi(a))^2. \]

We have \( \psi(c) > 0 \) for \( c < a_0 = 1.60645 \), \( \psi(a_0) = 0 \) and \( \psi(c) < 0 \) for \( c > a_0 \). Combining this fact with symmetry in the problem we obtain that \( (a_0, a_0) \) is the equilibrium point. Thus we can formulate the following

**Theorem 3.** For the non-zero-sum game on \([0, \infty)^2\) with the payoff functions (4.2) the equilibrium point exists and it is \((a, b)\) with \( a = b = a_0 \approx 1.6065 \) where \( a_0 \) is defined by \( \psi(c) = 0 \). The common equilibrium value is

\[ e^{-a_0} \Phi(a_0) - \frac{1}{2} (e^{-a_0} \Phi(a_0))^2 \approx 0.3830. \]

Therefore the probability of draw of the game is 0.2339.

Comparing the result obtained in Theorem 3, with that in Theorem 1(i) for the one-person game version, we observe

<table>
<thead>
<tr>
<th>One-person game:</th>
<th>a.d.l. is ( e^{-a_1} ) with ( a_1 \approx 1.5029 ) a.p.w. is 0.5174</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two-person game:</td>
<td>( e^{-a_0} ) with ( a_0 \approx 1.6065 ) 0.3830</td>
</tr>
</tbody>
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FULL-INFORMATION BEST-CHOICE PROBLEMS

REFERENCES


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