COMPETITIVE PREDICTION OF A RANDOM VARIABLE

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Abstract. Two players want to guess a realization of a random variable whose distribution is a priori known to both players. Each player gives his predicted value of the random variable. The winner is a player who has chosen the value greater than that chosen by his opponent but smaller than the realization of the random variable. Two models of this competitive prediction problem are formulated and solved — with common random variable for the players and with separate random variable for each one. Such games can be simple models of competition between players in some random games like stock market or card game. It is shown that there are several unsolved problems within or around this field.

1. Introduction. Two players, Player 1 and Player 2, compete to predict the realized value of a random variables \( \tau \), whose continuous distribution function is known to both players. The winner is a player, who has predicted the value not larger than and nearest to \( \tau \). He is paid by the loser a unit amount (in the zero-sum game version). The loser gets nothing if the game is non-zero-sum. Each player seeks to find the strategy that maximizes his expected reward.

The problem is closely related to the silent duel (see Karlin [4] and Sakaguchi [6]) and the auction bidding (see Sakaguchi [7]). In the cited papers the payoff functions are slightly different than those considered in this paper. Domanskiy [1] considered a similar problem for a sequence of Bernoulli trials, i.e. when \( \tau \) has the geometric distribution. Styszyński [9] investigated some related game model when players observe a Markov chain. The problem is also related to the two-person game approach to the parking problem of Sakaguchi and Tamaki [8] and the “game of boldness” by Henig and O’Neill [3]. The two person full-information best choice problem with imperfect observation, considered by Neumann, Porosiński and Szajowski [5], leads also to a very similar zero-sum game with the payoff function different than those in other mentioned problems.

In Section 2 the competitive prediction with common random variable for the players is discussed. The competitive prediction, when each player has separate random variable, is considered in Section 3. Optimal mixed strategies are derived in both cases. It is shown that the support of the optimal strategies in the case with common random variable for the players is larger than in the case with separate, but with the same distribution, random variable for each player. In Section 4 non-zero-sum game versions are formulated and solved. In the first of them re-prediction is allowed (§ 5.1) and in the second one the noisy prediction is adopted (§ 5.2). In the final Section 6 the cake-division game with a random termination time is discussed.

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2. Competitive prediction of a common random variable. Let \( \tau \) be a random variable defined on the probability space \((\Omega, \mathcal{F}, P)\) with the continuous distribution \(\Phi(x)\). Without loss of generality one can assume that the distribution is uniform on \([0, 1]\). Let \(x\) and \(y\) be players' predictions. Denoting by \(a\)-after \((b\)-before\) the event \(x > (\leq) \tau\) for Player 1, and similarly for Player 2 also, the result of the game is given by:

\[
\left\{ \begin{array}{ll}
a = a & \text{draw.} \\
a = b & \text{Player 2's win.} \\
b = a & \text{Player 1's win.} \\
b = b & \text{Player 1(2)'s win, if } x > (\leq) y; \text{ draw, if } x = y. \end{array} \right.
\]

It means that if \(b = b\) happens, the bolder player wins. If player's prediction is smaller than the realization and the opponent's prediction is greater than the realization of the random variable, then the player is winner, otherwise there is draw.

Player 1 gets from second player \(+1\) \((-1, 0)\) if the game is Player 1's win (Player 2's win, draw). The problem can be treated as the zero-sum game with the payoff function

\[
K(x, y) = \left\{ \begin{array}{ll}
-x + 2y - 1, & \text{if } x < y, \\
0, & \text{if } x = y, \\
-2x + y + 1, & \text{if } x > y. 
\end{array} \right.
\]

**Theorem 1.** For the competitive prediction of common continuous random variable with the rule described by (1), the related zero-sum game has the payoff (2). The optimal strategies of the players, \(f^*\) for Player 1 and \(g^*\) for Player 2, are mixed one having the continuous distribution with the density

\[
f^*(x) = g^*(x) = \left\{ \begin{array}{ll}
\frac{1}{2}(1 - x)^{-\frac{3}{2}}, & \text{if } 0 \leq x < \frac{3}{4}, \\
0, & \text{elsewhere.}
\end{array} \right.
\]

The value of the game is 0.

**Proof.** By standard method (see Karlin [4]) the common optimal randomized strategy with continuous distribution function \(F\) satisfies the differential equation

\[
\frac{F''(x)}{1 + F(x)} = \frac{1}{2(1 - x)}
\]

on the support of \(F\). Integration gives the optimal strategy. The support is determined by optimality conditions. \(\square\)

**Example 1.** In general, if \(\tau\) has the continuous distribution function \(H(x)\), then the payoff function is (2) with \(x\) and \(y\) replaced by \(u = H(x)\) and \(v = H(y)\), respectively. Hence the common optimal strategy has the continuous distribution function

\[
F^*(x) = G^*(x) = \left\{ \begin{array}{ll}
0, & \text{if } x < H^{-1}(0), \\
(1 - H(x))^{-\frac{1}{2}} - 1, & \text{if } H^{-1}(0) \leq x \leq H^{-1}(\frac{3}{4}), \\
1, & \text{if } x \geq H^{-1}(\frac{3}{4}).
\end{array} \right.
\]

**Example 2.** On the probability space \((\Omega, \mathcal{F}, P)\) a continuous time process \(X : T \times \Omega \rightarrow \mathbb{E} \subset \mathbb{R}\), where \(T \subset \mathbb{R}\), is defined. Let \(A \subset \mathbb{E}\) be (absorption states) such that \(\tau_A = \inf \{t \in T : X(t) \in A\}\) fulfills condition \(0 < P\{0 < \tau_A < \infty\} \leq 1\). There are two players, Player 1 and Player 2. They would like to predict the value of \(\tau_A\) based on knowledge about the distribution of the process only. They choose numbers \(u\) and \(v\), respectively, before the process is started. Player 1 is a winner if \(v < u \leq \tau_u\) or \(u \leq \tau_u < v\). Denote \(H(t) = P\{\tau_A \leq t\}\). Players play the zero sum game with the payoff function defined in Example 1.
Let $X$ be the Poisson process with parameter $\lambda$. Define $A = [1, \infty]$. The optimal strategy of Player 1 is mixed one with continuous distribution

$$F^*(u) = G^*(u) = \begin{cases} 
0, & \text{if } u < 0, \\
e^{\frac{1}{3}u} - 1, & \text{if } 0 \leq u < \frac{1}{3} \ln 4, \\
1, & \text{if } u \geq \frac{1}{3} \ln 4.
\end{cases}$$

3. Competitive prediction of an individual random variables. Let $\tau_1$ and $\tau_2$ be independent, identically distributed random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with the common continuous distribution $H(x)$. Without loss of generality one can assume that the distribution is uniform on $[0, 1]$. Let $x$ and $y$ be players' predictions. Denoting by $a$-after $(b$-before) the event $x > (<) \tau_i$ for Player $i, i = 1, 2$, the result of the game is given by:

$$(3) \quad \begin{array}{l}
a - a \\
b - a \\
b - b \text{ and } x \neq y \\
b - b \text{ and } x = y
\end{array} \begin{array}{l}
\text{happens, the game is} \\
\text{draw.} \\
\text{Player 2's win.} \\
\text{Player 1's win.} \\
\text{highest predictor's win.} \\
\text{Player 1(2)'}s \text{ win, if } \tau_1 < (> ) \tau_2.
\end{array}$$

Let $E_i(x, y), i = 1, 2, 3, 4,$ be the rectangular regions in $[0, 1] \times [0, 1]$, such that $E_1 = \{(t_1, t_2) \in [0, 1]^2 : 0 \leq t_1 \leq x, 0 \leq t_2 \leq y\}$, $E_2 = \{(t_1, t_2) \in [0, 1]^2 : x < t_1 \leq 1, 0 \leq t_2 \leq y\}$, $E_3 = \{(t_1, t_2) \in [0, 1]^2 : x < t_1 \leq 1, y < t_2 \leq 1\}$ and $E_4 = \{(t_1, t_2) \in [0, 1]^2 : 0 < t_1 < x, y < t_2 \leq 1\}$. Then under the rule (3), the conditional payoff given that $(\tau_1, \tau_2) = (t_1, t_2)$ is

$$(4) \quad k(x, y) = \begin{cases} 
0, & \text{if } (t_1, t_2) \in E_1(x, y), \\
1, & \text{if } (t_1, t_2) \in E_2(x, y), \\
\text{sgn}(x - y), & \text{if } x \neq y \text{ and } (t_1, t_2) \in E_3(x, y), \\
\text{sgn}(t_2 - t_1), & \text{if } x = y \text{ and } (t_1, t_2) \in E_3(x, y), \\
-1, & \text{if } (t_1, t_2) \in E_4(x, y).
\end{cases}$$

Therefore the expected payoff is

$$(5) \quad K(x, y) = \begin{cases} 
-xy + 2y - 1, & \text{if } x < y, \\
0, & \text{if } x = y, \\
xy - 2x + 1, & \text{if } x > y.
\end{cases}$$

We have

\textbf{Theorem 2.} For "the competitive prediction of each continuous random variables" with the rule described by (3), the related zero-sum game has the payoff (5). The optimal strategies of the players, $f^*$ for Player 1 and $g^*$ for Player 2, are mixed one having the continuous distribution with the density

$$f^*(x) = g^*(x) = \begin{cases} 
\frac{1}{4}(1 - x)^{-3}, & \text{if } 0 \leq x < \frac{2}{3}, \\
0, & \text{elsewhere}.
\end{cases}$$

The value of the game is 0.

\textit{Proof.} Standard method (see Karlin [4]) gives the differential equation

$$\frac{f'(x)}{f(x)} = \frac{3}{1 - x}$$

on the support of $F$. Integration gives the optimal strategy. The support is determined by optimality conditions.
Example 3. In general, if \( \tau_1 \) and \( \tau_2 \) has the continuous distribution function \( H(x) \), then the payoff function is (5) with \( x \) and \( y \) replaced by \( u = H(x) \) and \( v = H(y) \), respectively. Hence the common optimal strategy has the continuous distribution function

\[
F^*(x) = G^*(x) = \begin{cases} 
0, & \text{if } x < H^{-1}(0), \\
\frac{1}{2}[1 - (1 - H(x))^{-2} - 1], & \text{if } H^{-1}(0) \leq x \leq H^{-1}(\frac{2}{3}), \\
1, & \text{if } x \geq H^{-1}(\frac{2}{3}).
\end{cases}
\]

Example 4. Let \( X : T \times \Omega \rightarrow \mathbb{E} \subset \mathbb{R} \) and \( Y : T \times \Omega \rightarrow \mathbb{E} \subset \mathbb{R} \), where \( T \subset \mathbb{R} \), be two independent stochastic processes. Define \( A \subset \mathbb{E}, B \subset \mathbb{E} \) (absorption states) such that \( \tau_A = \inf\{t \in T : X(t) \in A\} \) and \( \tau_B = \inf\{t \in T : Y(t) \in B\} \) fulfill conditions \( 0 < P\{0 < \tau_A < \infty\} \leq 1 \). Player 1 would like to predict the moment \( \tau_A \) and Player 2 the moment \( \tau_B \). Let the first and second players choose \( u \) and \( v \), respectively, and \( u \neq v \). In this case Player 1 is a winner if \( (\tau_A \geq u, \tau_B \geq v, u > v) \) or \( (\tau_A > u, \tau_B \leq v) \) and Player 2 is a winner if \( (\tau_B \geq v, \tau_A \geq u, u < v) \) or \( (\tau_A \leq u, \tau_B > v) \). When both players choose the same value \( u \) then we have two cases. None of the process attained his absorption states. In this case the winner is the player having state of his process lower then opponent. If only one process attains his absorbing states the opponent is a winner. In other cases there is the tie.

Based on above description of the game the payoff \( K(u, v) \) function is derived.

\[
K(u, v) = \begin{cases} 
P\{\tau_A > u, \tau_B \leq v\} - P\{\tau_B > v\}, & \text{if } u < v, \\
\varphi(u), & \text{if } u = v, \\
P\{\tau_A > u\} - P\{\tau_A \leq u, \tau_B > v\}, & \text{if } u > v,
\end{cases}
\]

where

\[
\varphi(u) = P\{\tau_B \leq u < \tau_A\} + P\{X(u) < Y(u), \tau_A > u, \tau_B > u\} - P\{\tau_A \leq u < \tau_B\} - P\{X(u) > Y(u), \tau_A > u, \tau_B > u\}.
\]

Denote \( \Phi_A(t) = P\{\tau_A \leq t\} \) and \( \Phi_B(t) = P\{\tau_B \leq t\} \). We get

\[
\varphi(u) = (1 - \Phi_A(u))\Phi_B(u) - \Phi_A(u)(1 - \Phi_B(u)) + (1 - \Phi_A(u))(1 - \Phi_B(u))(1 - 2P\{X(u) - Y(u) > 0 \mid \tau_A > u, \tau_B > u\})
\]

and

\[
K(u, v) = \begin{cases} 
2\Phi_B(v) - \Phi_A(u)\Phi_B(v) - 1, & \text{if } u < v, \\
\varphi(u), & \text{if } u = v, \\
-2\Phi_A(u) + \Phi_A(u)\Phi_B(v) + 1, & \text{if } u > v.
\end{cases}
\]

The solution of the general case, when \( \Phi_A(u) \neq \Phi_B(u) \), is unknown. If \( X \) and \( Y \) have the same probability distributions and \( A = B \), then \( \varphi(u) = 0 \). Denote \( \Phi(t) = \Phi_A(t) = \Phi_B(t) \). In this case we have

\[
K(u, v) = \begin{cases} 
2\Phi(v) - \Phi(u)\Phi(v) - 1, & \text{if } u < v, \\
0, & \text{if } u = v, \\
-2\Phi(u) + \Phi(u)\Phi(v) + 1, & \text{if } u > v.
\end{cases}
\]

From Theorem 2 the following corollary follows:

**Corollary 1.** The two person zero-sum game related to the prediction of absorption moment for two independent, continuous time, identically distributed processes has the payoff
function (6). The equilibrium strategies of the players \((F^*(u), G^*(v))\) are mixed one, having
the continuous distribution

\[
F^*(u) = G^*(u) = \begin{cases} 
0, & \text{if } u < 0, \\
\frac{1}{8} \left[ (1 - \Phi(u))^{-2} - 1 \right], & \text{if } 0 \leq u < b^*, \\
1, & \text{if } u \geq b^*, 
\end{cases}
\]

where \(b^* = \inf \{ t > 0 : \Phi(t) = \frac{2}{3} \} \).

Let us assume that the Poisson processes \(X\) and \(Y\) with the same parameter \(\lambda\) are
observed. Define \(A = B = [1, \infty] \). The optimal strategy of Player 1 and Player 2 are mixed
one with continuous distribution

\[
F(u) = \begin{cases} 
0, & \text{if } u < 0, \\
\frac{1}{8}(e^{2\lambda u} - 1), & \text{if } 0 \leq u < \frac{1}{\lambda} \ln 3, \\
1, & \text{if } u \geq \frac{1}{\lambda} \ln 3.
\end{cases}
\]

Example 5. Let \((\tau_1, \tau_2)\) be distributed according to bivariate uniform with the probability
density function

\[
h(t_1, t_2) = 1 + \gamma(1 - 2t_1)(1 - 2t_2), \quad (t_1, t_2) \in [0, 1]^2, \quad |\gamma| \leq 1.
\]

Denote \(\bar{x} = 1 - x\). The payoff function becomes

\[
K(x, y) = \begin{cases} 
-xy + 2y - 1 - \gamma x\bar{x}y\bar{y}, & \text{if } x < y, \\
0, & \text{if } x = y, \\
x\bar{y} - 2x + 1 + \gamma x\bar{x}y\bar{y}, & \text{if } x > y,
\end{cases}
\]

since

\[
\int_{E_{i}(x, y)} h(t_1, t_2)dt_1dt_2 = \begin{cases} 
x(1 + \gamma \bar{x}y), & \text{if } i = 1, \\
\bar{x}(1 - \gamma x\bar{y}), & \text{if } i = 2, \\
\bar{x}y(1 + \gamma xy), & \text{if } i = 3, \\
x\bar{y}(1 - \gamma x\bar{y}), & \text{if } i = 4.
\end{cases}
\]

Although this is an important example in real life the solution is not known.

Remark 1. Comparing Theorem 2 with Theorem 1 we get an interesting observation. The
support of optimal strategies in the game with the common random variable is greater than
the support of the optimal strategies when players predict the individual, mutually
independent random variables with the same continuous distribution. The optimal probability
distribution function is more concentrated in the latter case.

Remark 2. If we suppose that \(\tau_1(\tau_2)\) be the time at which a certain accident may happen
to Player 1(2), then the rule (3) states that, when \(b - b\) happens, the bolder player wins.

The rule (3) in competitive prediction of the individual random variables does not seem
to be a natural analogue of (1) in the competitive prediction of common random variable.
If a natural analogue is to be considered, it should be

\[
(7) \quad \text{When} \begin{cases} 
a - a \\
a - b \\
b - a \\
b - b
\end{cases} \text{happens, the game is} \begin{cases} 
draw. \\
Player 2's \text{ win.} \\
Player 1's \text{ win.} \\
Player 1(2)'s \text{ win, if } \tau_1 - x < (>) \tau_2 - y.
\end{cases}
\]

To see why (3) is queer and (7) is natural, consider for example the strategy pair \((x, y) = (0.2, 0.3)\) and the event \((\tau_1, \tau_2) = (0.21, 0.99)\), then the game results in Player 2 [Player 1]'s
win by (3) [(7)] and moreover \(P\{0 < \tau_1 - 0.2 < \tau_2 - 0.3\} = 0.245\) is not small.
Under the modification (7), the payoff becomes (4) with $\text{sgn}(x - y)$ replaced by $\text{sgn}(t_2 - y - t_1 + x)$, so that the expected payoff function is

$$K(x, y) = \begin{cases} 
  y^2 - xy, & \text{if } x < y, \\
  0, & \text{if } x = y, \\
  -x^2 + xy, & \text{if } x > y.
\end{cases}$$

(8)

Unfortunately, however, we easily find

**Proposition 1.** The zero-sum game with payoff function (8) has a saddle point $(x^*, y^*) = (0, 0)$ and the saddle value 0.

If one player wishes to make sure of his success and expects his opponent’s failure, then the same is for his opponent, in the optimal play.

4. **Non-zero-sum version of prediction.** In order to dissolve this perplexity, described in Remark 2 a conjecture may arise that non-zero-sum game version will be useful. However, we find that situation remains unchanged. Theorems in this section are stated without proof. Proofs are easy.

**Theorem 3 (Non-zero-sum game / rule (1)).** For “the competitive prediction of common random variable” with payoff functions

$$(K_1(x, y), K_2(x, y)) = \begin{cases} 
  (y - x, 1 - y), & \text{if } x < y, \\
  (0, 0), & \text{if } x = y, \\
  (1 - x, x - y), & \text{if } x > y,
\end{cases}$$

the equilibrium strategies for both players are the same with the density function

$$f^*(x) = g^*(x) = \begin{cases} 
  (1 - x)^{-1}, & \text{if } 0 < x < 1 - e^{-1} \approx 0.6321, \\
  0, & \text{elsewhere.}
\end{cases}$$

The equilibrium values are $(v_1, v_2) = (e^{-1}, e^{-1})$ and $P\{\text{draw}\} = 1 - 2e^{-1}$.

The extension of this non-zero-sum game to $n$-person $(n \geq 3)$ version is discussed in Sakaguchi [7], Section 4, and it is shown that deriving the explicit solution is not easy in spite of its easy formulation of the problem.

**Theorem 4 (Non-zero-sum game / rule (3)).** For “the competitive prediction of each random variable” with payoff functions

$$(K_1(x, y), K_2(x, y)) = \begin{cases} 
  ((1 - x)y, 1 - y), & \text{if } x < y, \\
  \left(\frac{1}{2}(1 - x^2), \frac{1}{2}(1 - x^2)\right), & \text{if } x = y, \\
  (1 - x, x(1 - y)), & \text{if } x > y,
\end{cases}$$

the equilibrium strategies for both players are the same with the density function

$$f^*(x) = g^*(x) = \begin{cases} 
  (\sqrt{2} - 1)(1 - x)^{-3}, & \text{if } 0 < x < 2 - \sqrt{2} \approx 0.5858, \\
  0, & \text{elsewhere.}
\end{cases}$$

The equilibrium values are $(v_1, v_2) = (\sqrt{2} - 1, \sqrt{2} - 1)$ and $P\{\text{draw}\} = 3 - 2\sqrt{2} \approx 0.1716$.

The extension of this game to $n$-person $(n \geq 3)$ version is not yet done. Again we have, unfortunately
Theorem 5 (Non-zero-sum game / rule (7)). For "the competitive prediction of each random variables", when the rule (3) is replaced by (7), then the payoff functions for the two-person case become

\[
(K_1(x, y), K_2(x, y)) = \begin{cases} 
\left( \frac{1}{2}(1 + y^2) - xy, \frac{1}{2}(1 - y^2) \right), & \text{if } x < y, \\
\left( \frac{1}{2}(1 - x^2), \frac{1}{2}(1 - x^2) \right), & \text{if } x = y, \\
\left( \frac{1}{2}(1 - x^2), \frac{1}{2}(1 + x^2) - xy \right), & \text{if } x > y.
\end{cases}
\]

This game has an equilibrium point \((x^*, y^*) = (0, 0)\), and the equilibrium value is \((v_1, v_2) = (0.5, 0.5)\).

5. Variation of the competitive prediction problem. In the present section we consider two diversion of our discussion. Suppose that Player 1(2) is waiting for a girl \(O_1(O_2)\) to have a date. Each girl \(O_1(O_2)\) appears in the date spot at time \(\tau_1(\tau_2)\). \(\tau_1\) and \(\tau_2\) are independent, identically distributed random variables with common, the uniform distribution on \([0, 1]\). Players 1(2) privately predicts the time at which his girl appears.

5.1. Case when re-prediction is allowed. In this case each player is informed of which of \(a\) and \(b\) has happened immediately after his prediction time has passed. If the outcome is \(b\), Player 1(2) modifies his prediction \(x\) to \(2x - x^2\) (\(y\) to \(2y - y^2\)). Each player knows that his opponent is allowed to re-predict in such a manner.

Then the conditional payoff under rule (3), given \((\tau_1, \tau_2) = (t_1, t_2)\), is

\[
k(x, y) = \begin{cases} 
0, & \text{if } (t_1, t_2) \in E_1(2x - x^2, 2y - y^2), \\
1, & \text{if } (t_1, t_2) \in E_2(2x - x^2, 2y - y^2), \\
\text{sgn}(2x - x^2 - 2y + y^2), & \text{if } x \neq y \text{ and } (t_1, t_2) \in E_3(2x - x^2, 2y - y^2), \\
\text{sgn}(t_2 - t_1), & \text{if } x = y \text{ and } (t_1, t_2) \in E_3(2x - x^2, 2y - y^2), \\
-1, & \text{if } (t_1, t_2) \in E_4(2x - x^2, 2y - y^2),
\end{cases}
\]

and so the expected payoff is

\[
K(x, y) = \begin{cases} 
x^2 - y^2 - (\bar{x}\bar{y})^2, & \text{if } 2x - x^2 < 2y - y^2, \\
0, & \text{if } 2x - x^2 = 2y - y^2, \\
x^2 - y^2 + (\bar{x}\bar{y})^2, & \text{if } 2x - x^2 > 2y - y^2.
\end{cases}
\]

We have

Theorem 6. For the zero-sum game with payoff function (9), the optimal strategy \(f^*\) for Player 1 and \(g^*\) for Player 2, are mixed one having the continuous distribution with the density

\[
f^*(x) = g^*(x) = \begin{cases} 
\frac{1}{2}(1 - x)^{-5}, & \text{if } 0 \leq x < 1 - \frac{1}{\sqrt{3}} \approx 0.4227, \\
0, & \text{elsewhere}.
\end{cases}
\]

The value of the game is 0.

Proof. Let \(\bar{x}^2 = \xi\) and \(\bar{y}^2 = \eta\). Then the payoff function (9) is rewritten as

\[
K(\xi, \eta) = \begin{cases} 
\xi - \eta + \xi \eta, & \text{if } \xi < \eta, \\
0, & \text{if } \xi = \eta, \\
\xi - \eta - \xi \eta, & \text{if } \xi > \eta,
\end{cases}
\]
where \((\xi, \eta) \in [0, 1]^2\). This payoff function of the transformed game is the payoff function for the silent duel with one bullet for each player (Karlin [4], pp. 136–140). This game has the value 0 and the common optimal strategy

\[
f^*(\xi) = g^*(\eta) = \begin{cases} \frac{1}{4} \xi^{-3}, & \text{if } \frac{1}{5} \leq \xi \leq 1, \\ 0, & \text{otherwise.} \end{cases}
\]

Turning the variable from \(\xi\) to \(x\) we get (10).

We observe that the support of optimal strategies in the game is smaller than the support of the optimal strategies given in Theorem 2.

Next we shall consider the case where re-prediction is allowed for Player 1 only. Player 2 knows that his opponent is allowed to re-predict. Then the expected payoff of the game under rule (3) is shown to be

\[
K(\xi, \eta) = \begin{cases} \bar{x}^2 - \bar{y} + \bar{x}^2 \text{sgn}(2x - x^2 - y), & \text{if } 2x - x^2 \neq y, \\ 0, & \text{otherwise.} \end{cases}
\]

We have

**Theorem 7.** For the zero-sum game with payoff function (12), the optimal strategy \(f^*\) for Player 1 is

\[
f^*(x) = \begin{cases} \frac{1}{2}(1 - x)^{-5}, & \text{if } 0 \leq x < 1 - \frac{1}{\sqrt{3}} \approx 0.4227, \\ 0, & \text{elsewhere,} \end{cases}
\]

and \(g^*\) for Player 2 is

\[
g^*(y) = \begin{cases} \frac{1}{4}(1 - y)^{-3}, & \text{if } 0 \leq y < \frac{2}{3}, \\ 0, & \text{elsewhere.} \end{cases}
\]

The value of the game is 0.

**Proof.** Let \(\bar{x}^2 = \xi\) and \(1 - y = \eta\). Then (12) reduces to (11) again. So we obtain the result.

Our intuition that the game is in favor of Player 1, and therefore the game value is positive, is wrong. That game (12) is an example of the continuous game on the unit square in which the two regions are separated by a curve other than the diagonal segment.

5.2. **Case when prediction is noisy.** Suppose that a player who predicts the time earlier than his opponent is requested to announce the outcome (i.e. which of the event \(a\) or \(b\) has happened) at his prediction time, so that his opponent reacts instantaneously to his information. If both players predict the same time, both must announce the outcomes. The reasonable reaction of each player, when his opponent’s announcement is made, is given in Table 1 (i.e. conditional payoff, given \((\tau_1, \tau_2) = (t_1, t_2)\), under rule (3), is given by Figure 1 - cf. \(P_1 = (x, y)\), \(P_2 = (2x - x^2, x + y - xy)\) and \(P_2' = (x + y - xy, 2y - y^2)\)).

**Table 1.** The reaction of the players

<table>
<thead>
<tr>
<th>Outcome (x \land y)</th>
<th>(a)</th>
<th>(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>if (x &lt; y) Player 1</td>
<td>keep (x)</td>
<td>change to (2x - x^2)</td>
</tr>
<tr>
<td>Player 2</td>
<td>change to (x + \epsilon)</td>
<td>change to (x + y - xy)</td>
</tr>
<tr>
<td>if (x &gt; y) Player 1</td>
<td>change to (y + \epsilon)</td>
<td>change to (x + y - xy)</td>
</tr>
<tr>
<td>Player 2</td>
<td>keep (y)</td>
<td>change to (2y - y^2)</td>
</tr>
<tr>
<td>if (x = y)</td>
<td>game ends with payoff (0, 1, \text{sgn}(\tau_2 - \tau_1), -1)</td>
<td>if (a - a, b - a, b - b, a - b), respectively.</td>
</tr>
</tbody>
</table>
Therefore the payoff function is

\[ K(x, y) = \begin{cases} 
-\bar{x}\bar{y} + \bar{x}^2(1 - \bar{y}) - x\bar{y}, & \text{if } x < y, \\
0, & \text{if } x = y, \\
\bar{x}\bar{y} - (1 - \bar{x})\bar{y}^2 + x\bar{y}, & \text{if } x > y,
\end{cases} \]

i.e. the game is symmetric. The value of the game is zero, but the optimal strategies are not known.

**Figure 1. The gain of the players**

6. **A cake-division game.** In the final section we consider a cake-division problem which is modelled by means of a non-zero-sum game of timing with a random duration time. Suppose that two players divide a cake of size 1 before a prescribed random termination time \( \tau \) comes. Initially Player 1(2) has his right to receive the amount \( \alpha_1(\alpha_2) \), where \( 0 < \alpha_1, \alpha_2 < 1 \) and \( \alpha_1 + \alpha_2 < 1 \). Player \( i, i = 1, 2 \), must choose a point in time \( t_i \in [0, \infty) \) to claim his piece of the cake. If \( t_1 < t_2 < \tau < (t_1 < \tau < t_2) \), Player 1 gets the discounted part \( \alpha_1 \delta^{t_1} \) of the cake, while Player 2 receives the discounted remaining part \( (1 - \alpha_1) \delta^{t_2} \), with \( 0 < \delta \leq 1 \). (Player 2 gets nothing.) For \( t_1 > t_2 \) the cake is divided in an analogous way, and if \( t_1 = t_2 < (>) \tau \), then each Player receives his discounted right and they share the remaining part equally (both players get nothing).

The above described game is "a silent duel over a cake" considered by Hamers [2] combined with a random duration time. We find solution of this game and make comparison with result of Hamers [2]. Let us assume that \( \tau \) has the uniform distribution on \([0, 1]\). When Player 1(2) has chosen a point in time \( t_1(t_2) \in [0, 1] \) to claim his piece of cake, the payoff functions are given by

\[ (K_1(t_1, t_2), K_2(t_1, t_2)) = \begin{cases} 
(\delta^{t_1} \bar{t}_1 \alpha_1, \delta^{t_2} \bar{t}_2 \alpha_1), & \text{if } t_1 < t_2, \\
(\delta^{t_1} \bar{t}_1 (\alpha_1 + \frac{1}{2}(1 - \alpha_1 - \alpha_2)), \delta^{t_2} \bar{t}_2 (\alpha_2 + \frac{1}{2}(1 - \alpha_1 - \alpha_2))), & \text{if } t_1 = t_2, \\
(\delta^{t_1} \bar{t}_1 \alpha_2, \delta^{t_2} \bar{t}_2 \alpha_2), & \text{if } t_1 > t_2.
\end{cases} \]

Note that in this game each player must choose his decision timing the earlier the better (since discounting and random termination are considered), but at the same time, later than his opponent (since \( \alpha_1 + \alpha_2 < 1 \) is imposed).

It is not difficult to see that there is no Nash equilibrium in pure strategies. So we shall derive a Nash equilibrium in mixed strategies. Supposing that \( F_i(t) \), \( i = 1, 2 \), consists of a density part \( f_i(t) > 0 \), \( i = 1, 2 \), over an interval \([0, a]\) and a mass part \( q_i \) at \( t = 0 \), we have for \( 0 < t < a \),

\[ K_1(t, F_2) = \delta^t (1 - t) \left[ q_2(1 - \alpha_2) + (1 - \alpha_2) \int_0^t f_2(y)dy + \alpha_1 \int_t^a f_2(y)dy \right] = n_1, \]
(16) \( K_2(F_1, t) = \delta^t (1 - t) \left[ q_1 (1 - \alpha_1) + (1 - \alpha_1) \int_0^t f_1(x) dx + \alpha_2 \int_t^a f_1(x) dx \right] \equiv \eta_2, \)

where \( \eta_1 \) and \( \eta_2 \) are equilibrium values.

**Theorem 8.** For the non-zero-sum game with payoff function (14), the equilibrium in mixed strategies is given by

(i) If \( \alpha_1 \geq \alpha_2 \), then

\[
\begin{align*}
f_1^*(t) &= \frac{\alpha_1 \bar{\alpha}_1}{\bar{\alpha}_2 (1 - \alpha_1 - \alpha_2)} \{ (1 - t)^{-2} - (\log \delta)(1 - t)^{-1} \} \delta^{-t}, \quad 0 < t < a, \\
with q_1 &= (\alpha_1 - \alpha_2)/(1 - \alpha_2), \\
f_2^*(t) &= \frac{\alpha_1}{1 - \alpha_1 - \alpha_2} \{ (1 - t)^{-2} - (\log \delta)(1 - t)^{-1} \} \delta^{-t}, \quad 0 < t < a,
\end{align*}
\]

where \( a \) is a unique root in \((0,1)\) of the equation \( \delta^a (1 - a) = \alpha_1 / \bar{\alpha}_2 \). Equilibrium values are \((\eta_1, \eta_2) = (1, \bar{\alpha}_1 / \bar{\alpha}_2) \alpha_1\).

(ii) If \( \alpha_1 < \alpha_2 \), then

\[
\begin{align*}
f_1^*(t) &= \frac{\alpha_2}{1 - \alpha_1 - \alpha_2} \{ (1 - t)^{-2} - (\log \delta)(1 - t)^{-1} \} \delta^{-t}, \quad 0 < t < a, \\
f_2^*(t) &= \frac{\alpha_2 \bar{\alpha}_2}{\bar{\alpha}_1 (1 - \alpha_1 - \alpha_2)} \{ (1 - t)^{-2} - (\log \delta)(1 - t)^{-1} \} \delta^{-t}, \quad 0 < t < a,
\end{align*}
\]

with \( q_2 = (\alpha_2 - \alpha_1)/(1 - \alpha_1) \) and where \( a \) is a unique root in \((0,1)\) of the equation \( \delta^a (1 - a) = \alpha_2 / \bar{\alpha}_1 \). Equilibrium values are \((\eta_1, \eta_2) = (\bar{\alpha}_2 / \bar{\alpha}_1, 1) \alpha_2\).

**Proof.** Differentiation of the equation (15) with respect to \( t \) gives

\[
\frac{(1 - \alpha_1 - \alpha_2) f_2(t)}{q_2 (1 - \alpha_2) + (1 - \alpha_2) \int_0^t f_2(y) dy + \alpha_1 \int_t^a f_2(y) dy} = (1 - t)^{-1} - \log \delta,
\]

which, combined with (15), yields

\[
f_2(t) = \frac{\eta_1}{1 - \alpha_1 - \alpha_2} \{ (1 - t)^{-2} - (\log \delta)(1 - t)^{-1} \} \delta^{-t}.
\]

The normalization condition \( \int_0^a f_2(t) dt = 1 - q_2 \) and the conditions \( K_1(0 + 0, F_2) = K_1(a - 0, F_2) = \eta_1 \) in (15) give

(17) \[ q_2 = \frac{\eta_1 - \alpha_1}{1 - \alpha_1 - \alpha_2}, \text{ and } \eta_1 = (1 - \alpha_2) \delta^a (1 - a). \]

Starting from the equation (16) and proceeding in an analogous way we find that

\[
f_1(t) = \frac{\eta_2}{1 - \alpha_1 - \alpha_2} \{ (1 - t)^{-2} - (\log \delta)(1 - t)^{-1} \} \delta^{-t},
\]

(18) \[ q_1 = \frac{\eta_2 - \alpha_2}{1 - \alpha_1 - \alpha_2}, \text{ and } \eta_2 = (1 - \alpha_1) \delta^a (1 - a). \]

In order to obtain the equilibrium strategies we shall determine parameter \( a \) which satisfies \( q_1 q_2 = 0 \). When \( q_2 = 0 \), (17) gives \( \eta_1 = \alpha_1 \) and \( \delta^a (1 - a) = \alpha_1 / \bar{\alpha}_2 \), and hence (18) gives \( \eta_2 = \alpha_1 \bar{\alpha}_1 / \bar{\alpha}_2 \) and \( q_1 = (\alpha_1 - \alpha_2)/(1 - \alpha_2) \). In addition to (15)–(16), we find

\[ K_1(t, F_2) = \begin{cases} 
(\alpha_1 + \frac{1}{2} (1 - \alpha_1 - \alpha_2)) q_2 + (1 - q_2) \alpha_1, & \text{if } t = 0 \\
\delta^t \bar{\alpha}_2, & \text{if } a \leq t \leq 1
\end{cases} \]
\[ K_2(F_1,t) = \begin{cases} 
(\alpha_2 + \frac{1}{2}(1 - \alpha_1 - \alpha_2))q_1 + (1 - q_2)\alpha_2, & \text{if } t = 0 \\
\delta^t\bar{\alpha}_1, & \text{if } a \leq t \leq 1 
\end{cases} \]

and therefore with these \( q_1, q_2 \) and \( a \) we find that, when \( q_2 = 0 \),

\[ K_1(t, F_2) = \alpha_1, \text{ if } t = 0; \quad \leq \delta^a\bar{\alpha}_2 = \alpha_1, \text{ if } a \leq t \leq 1 \]

\[
K_1(F_1, 0) = (\alpha_2 + \frac{1}{2}(1 - \alpha_1 - \alpha_2))(\alpha_1 - \alpha_2)/\bar{\alpha}_2 + (\bar{\alpha}_1/\bar{\alpha}_2)\alpha_2 \\
= \frac{1}{2}(\alpha_2 + \alpha_1\bar{\alpha}_1/\bar{\alpha}_2) \\
\leq \alpha_1\bar{\alpha}_1/\bar{\alpha}_2, \quad \text{if } \alpha_1 \geq \alpha_2
\]

and

\[ K_2(F_1, t) = \delta^t\bar{\alpha}_1 \leq \delta^a\bar{\alpha}_1 = \alpha_1\bar{\alpha}_1/\bar{\alpha}_2, \text{ if } a \leq t \leq 1 \]

This is the case (i) mentioned in the theorem.

When \( q_1 = 0 \), (18) gives \( \eta_2 = \alpha_2 \) and \( \delta^a(1-a) = \alpha_2/\bar{\alpha}_1 \) and hence (17) gives \( \eta_1 = \alpha_2\bar{\alpha}_2/\bar{\alpha}_1 \) and \( q_2 = (\alpha_2 - \alpha_1)/(1 - \alpha_1) \). Therefore with these \( q_1, q_2 \) and \( a \) we analogously get

\[ k_1(0, F_2) \leq \alpha_2\bar{\alpha}_2/\bar{\alpha}_1, \text{ if } \alpha_1 \leq \alpha_2 \]

\[ K_1(t, F_2) = t\bar{\alpha}_2 \leq \delta^a\bar{\alpha}_2 = \alpha_2\bar{\alpha}_2/\bar{\alpha}_1, \text{ if } a \leq t \leq 1 \]

and

\[ K_2(F_1, t) = \alpha_2, \text{ if } t = 0; \quad \leq \delta^t\bar{\alpha}_1 = \alpha_2, \text{ if } a \leq t \leq 1. \]

It thus follows that in both cases of (i) and (ii) \((F_1, F_2)\) constitutes an equilibrium, i.e.,

\[
\max_{F} K_1(F_1, F_2) = K_1(F_1, F_2) = \eta_1 \\
\max_{F} K_2(F_1, F_2) = K_2(F_1, F_2) = \eta_2.
\]

This complete the proof of the theorem. \( \Box \)

Remark 3. In order to obtain the equilibrium, the “weak” player (i.e. that one whose right is smaller than his opponent’s) should distribute his search for his decision-timing without any empty spot, whereas the “strong” player is able to possess a positive probability mass at time 0. Moreover, the strong player can get the amount exactly equal to his right, whereas the weak player can get the amount lying between his own and his rival’s rights.

Remark 4. Our Theorem 8 and the result in Hamers [2] show that the equilibrium values remain unchanged independently of whether random termination of the game is considered or not.

Remark 5. Two special cases are to be noted.

\(^1 \) When \( \alpha_1 + \alpha_2 = 1 \), (i.e. the rights of the players constitute a division of the whole cake), each player obviously will claim his right at time 0, and gets his own right \( \alpha_i \).
When \( \delta = 1 \), (i.e. no-discounting) we have, for \( a = (1 - \alpha_1 - \alpha_2)/\bar{\alpha}_2 \),

\[
\begin{align*}
    f_1'(t) &= \frac{\alpha_1 \bar{\alpha}_1}{\bar{\alpha}_2 (1 - \alpha_1 - \alpha_2)} (1 - t)^{-2}, \quad q_1 = \frac{\alpha_1 - \alpha_2}{1 - \alpha_2}, \\
    f_2'(t) &= \frac{\alpha_1}{1 - \alpha_1 - \alpha_2} (1 - t)^{-2}, \quad q_2 = 0,
\end{align*}
\]

if \( \alpha_1 \geq \alpha_2 \). And similar result if \( \alpha_1 < \alpha_2 \).

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