OPTIMAL STOPPING OF RANDOM FIELDS.  
PROBABILITY MAXIMIZING APPROACH

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Abstract. The paper deals with the optimal stopping problem for non-negative random fields. Necessary and sufficient conditions for existence of optimal strategy are given. The results are used to solve the probability maximizing version of the optimal stopping problem for random fields indexed by a countable partially ordered set. We specialize our results to the problem of optimal stopping for several Markov chains. Examples concerning the problem of optimal allocation of different treatments and some problems of optimal selection are given.

1. Introduction. The subject of the paper is the optimal stopping problem for discrete multi-parameter processes (random fields) to guarantee the maximal probability of achieving some set. The result is preceded by necessary and sufficient conditions for existence of an optimal strategy in the optimal stopping problem for the non-negative random fields. In 1966 Haggstrom [6] generalized Snell's results to processes indexed by tree. His results are particular case of the general theory of optimal stopping for a family of random variables indexed by a parameter from a partially ordered set. In 1980 Krengel and Sucheston [8],[9] proved a number of theorems developing the general theory. Among other things they introduced the notion of tactic and defined the qualitative conditional independence (CQI) under which every stopping rule is given by a tactic. Further, Mandelbaum and Vanderbei [10] generalized Snell's result to the case of discrete multi-parameter processes indexed by a parameter from a partially ordered set. They focused their considerations on the stopping points belonging to some subclass called the predictable stopping points. The results were applied to the problem of optimal stopping of several independent Markov chains. Recently, Nualart [13] has showed the existence of optimal stopping points for upper semicontinuous two-parameter processes in continuous time.

The present paper is based on Mandelbaum and Vanderbei's results. Following Bojdecki [1],[2] we develop the probability maximizing approach to the problem of optimal stopping for the random fields indexed by a countable partially ordered set (see also [18]). We prove a theorem which gives sufficient and necessary conditions for existence of optimal strategy for non-negative random fields. As in Mandelbaum and Vanderbei we restrict ourselves to the class of predictable stopping points. We also prove a theorem on optimal stopping for Markov random fields. Further, the examples of application of the theorems are given.

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The paper is organized as follows. In Section 2 we formulate the results for non-negative random fields. Next, the probability maximizing approach to the problem of optimal stopping for random fields indexed by a countable partially ordered set is formulated and its solution is got from the results for the non-negative random fields. In Section 3 we discuss the problem of optimal stopping for Markov random fields. The main result of the section is a modification of similar result for stochastic processes in Neveu [12]. In Section 4 we develop Mandelbaum and Vanderbei's results concerning the problem of optimal stopping for several Markov chains. An example concerning the problem of optimal allocation of different treatments (cf. [5]) is given. In Section 5 we discuss some problem of optimal selection (see [14] or [4] for references) connected with buying an asset in different places or with selecting a product from the different factories. We give also some numerical examples and asymptotic considerations for the selection model.

2. Optimal stopping of random fields. A condition of optimality and probability maximizing approach. Let \( S \) be a partially ordered countable set. We assume that there exists a unique element \( o \in S \) such that \( o \preceq s \) for each \( s \in S \). By \([r, s]\) we denote the set of all elements of \( u \in S \) for which \( r \preceq u \preceq s \). We say that a point \( s \in S \) is a direct successor of \( r \) in \( S \) if there exists no point \( u \in S \) such that \( r \prec u \prec s \). In other words - a point \( s \) is a direct successor of \( r \) if the set \([r, s]\) consists of only two points \( r \) and \( s \). The set of direct successors of a point \( s \) will be denoted by \( U(s) \). We assume that for each \( s \in S \) the set \( U(s) \) has only finite number of elements. We also assume that for each \( r \preceq s \) the set \([r, s]\) is finite. The element of \( S \) which is the successor of each element will be denoted by \( \infty \).

Let \((\Omega, \mathcal{F}, P)\) be a probability space. By \( \mathcal{F}_s \) we denote a family of \( \sigma \)-sub-algebras of \( \mathcal{F} \) indexed by \( s \in S \) such that \( \mathcal{F}_r \subseteq \mathcal{F}_s \) for \( r \preceq s \). A random variable \( \xi \) with values in \( S \) is called a stopping point if for each \( s \in S \) we have \( \{ \xi \preceq s \} \in \mathcal{F}_s \) which is equivalent to the condition that for each \( s \in S \) we have \( \{ \xi = s \} \in \mathcal{F}_s \). The class of all stopping points will be denoted by \( \Xi \).

Following Mandelbaum and Vanderbei's [10] paper we restrict ourselves to the set of predictable stopping points. Let us recall the definition of the set of predictable stopping points.

Let \( \sigma_t, t = 0, 1, 2, \ldots \) be an increasing sequence of stopping points such that

\[
\sigma_t \preceq \sigma_{t+1} \quad \text{for each } t \\
\sigma_{t+1} \in \mathcal{F}_{\sigma_t} = \mathcal{G}_t \quad \text{i.e. } \sigma_{t+1} \text{ is } \mathcal{G}_t \text{ measurable}.
\]

The sequence \( \sigma_t, t = 0, 1, 2, \ldots \) will be called a predictable strategy. The strategy is continuous if \( \sigma_{t+1} \) is a direct successor of \( \sigma_t \) for each \( t \). In the sequel, the sequence \( \sigma_t, t = 0, 1, 2, \ldots \) will be denoted by \( \sigma \) in short. A pair \( \pi = (\sigma_t, \tau) \) consisting of a continuous strategy \( \sigma_t, t = 0, 1, 2, \ldots \) and a stopping time \( \tau \) with respect to the family \( \{\mathcal{G}_t\} \) is said to be a policy. The set of all policies \( \pi = (\sigma_t, \tau) \) for which \( \sigma_0 = 0 \) will be denoted by \( \Pi \). To every policy \( \pi = (\sigma_t, \tau) \in \Pi \) corresponds a stopping point \( \alpha(\pi) = \sigma_\tau \) with respect to \( \{\mathcal{F}_s\}_{s \in S} \). Stopping points defined in this way are called predictable. More details about the class of stopping points one can find in Mandelbaum and Vanderbei [10] and Walsh [17].

Let \( \{Y_s\}_{s \in S} \) be a family of non-negative, adapted to \( \mathcal{F}_s \), random variables. We say that \( \{Y_s\}_{s \in S} \) is of class \( \mathcal{D} \) if the family \( \{Y_\xi\}_{\xi \in \Xi} \) is uniformly integrable. For instance, if \( \sup_{s \in S} Y_s < \infty \) then \( \{Y_s\}_{s \in S} \) is of class \( \mathcal{D} \). In the theory of optimal stopping the random variable \( \{Y_s\} \) is interpreted as a reward obtained after stopping an observation at \( s \). The reward obtained in the case if we never stop is denoted by \( Y_{\infty} \) and it is assumed to be a random variable. It is worth noting that we do not make any assumption about the existence of a special \( \infty \) point in the set \( S \). Further we will assume that \( Y_{\infty} = 0 \) which
merely means that to get anything we should stop. In the mentioned paper of Mandelbaum and Vanderbei it was found an optimal policy \( \pi^* = (\sigma^*_t, \tau^*) \) in the sense that

\[
EY_{\alpha(\pi^*)} = \sup_{\pi \in \Pi} EY_{\alpha(\pi)}.
\]

Other sufficient conditions for existence of optimal policy are given in Mazziotto and Szpirglas [11].

The theorem which we are to prove gives a necessary and sufficient condition, under some assumption about \( Y_s \), for a policy to be optimal. Let

\[
X_s = \text{ess sup}_{\alpha(\pi) \in \Pi_s} E(Y_{\alpha(\pi)} \mid \mathcal{F}_s)
\]

where \( \Pi_s \) denotes the set of predictable continuous strategies for which \( \sigma_0 = s \). For a predictable strategy \( \sigma_t, t = 0, 1, 2, \ldots \), let us define

\[
\tau^0(\sigma) = \inf\{ t : X_{\sigma_t} = Y_{\sigma_t} \}.
\]

**Theorem 1.** Let us assume that \( Y_s, s \in S \) is of class \( \mathcal{D} \) and \( Y_\infty = 0 \). If a predictable strategy \( \sigma_t, t = 0, 1, 2, \ldots \) is such that \( \limsup_{t \to \infty} Y_{\sigma_t} = 0 \) on the set \( \{ \tau^0(\sigma) = \infty \} \) then \( \pi = (\sigma_t, \tau^0(\sigma)) \) is an optimal predictable strategy.

**Theorem 2.** If \( \pi' = (\sigma^*_t, \tau') \) is an optimal predictable strategy then we have \( \limsup_{t \to \infty} Y_{\sigma^*_t} = 0 \) on the set \( \{ \tau^0(\sigma') = \infty \} \), where the process \( Y_s \) satisfies the same assumptions as in previous theorem.

Before we prove the theorems we need some lemmas.

**Lemma 1.** Let \( \rho^1 = (\sigma^1_t, \tau_1), \alpha(\rho_1) = \sigma^1_{\tau_1} \). By \( \Pi_{\rho_1} \) we denote the set of all predictable strategies \( \pi = (\sigma_t, \tau) \) for which \( \alpha(\pi) = \sigma_\tau \geq \alpha(\rho_1) = \sigma^1_{\tau_1} \). Then

\[
X_{\alpha(\rho_1)} = \text{ess sup}_{\alpha(\pi) \in \Pi_{\rho_1}} E(Y_{\alpha(\pi)} \mid \mathcal{F}_{\alpha(\rho_1)})
\]

**Proof.** Let \( \pi \in \Pi_{\rho_1} \).

\[
E(Y_{\sigma_{\tau}} \mid \mathcal{F}_{\sigma^1_{\tau_1}}) = \sum_u \mathbb{I}_{\{\sigma^1_{\tau_1} = u\}} E(Y_{\sigma_{\tau}} \mid \mathcal{F}_u) = \sum_u E(\mathbb{I}_{\{\sigma^1_{\tau_1} = u\}} Y_{\sigma_{\tau}} \mid \mathcal{F}_u)
\]

We now know that on the set \( \{\sigma^1_{\tau_1} = u\} \) we have \( \sigma_\tau \geq u \). Let us define the following predictable strategy

\[
\tilde{\pi} = (\tilde{\sigma}_t, \tau) = (\sigma_t \vee u, \tau).
\]

Thus \( \tilde{\pi} \in \Pi_u \) and \( \tilde{\sigma}_t = \sigma_\tau \vee u \). We see that on the set \( \{\sigma^1_{\tau_1} = u\} \) we have \( \tilde{\sigma}_t = \sigma_\tau \vee u = \sigma_\tau \) since \( \sigma_\tau \in \Pi_{\rho_1} \) and \( \sigma_\tau \geq \sigma^1_{\tau_1} \).

Thus we can write

\[
\sum_u E(\mathbb{I}_{\{\sigma^1_{\tau_1} = u\}} Y_{\sigma_{\tau}} \mid \mathcal{F}_u) = \sum_u E(\mathbb{I}_{\{\sigma^1_{\tau_1} = u\}} Y_{\tilde{\sigma}_t} \mid \mathcal{F}_u) = \sum_u \mathbb{I}_{\{\sigma^1_{\tau_1} = u\}} X_u = X_{\sigma^1_{\tau_1}}.
\]
To prove the reversed inequality let us note that

$$X_{\sigma_1^1} = \sum_u I_{\{\sigma_1^1 = u\}} X_u = \sum_u I_{\{\sigma_1^1 = u\}} \text{ess sup}_{\pi \in \Pi_u} E(Y_{\alpha(\pi)} \mid \mathcal{F}_u).$$

Let $\pi \in \Pi_u$, $\pi = (\sigma_t, \tau)$. Introducing the strategy $\pi' = (\beta_t, \gamma)$, where

$$\beta_t = \sigma_t I_{\{\alpha(\rho_t) = u\}} + (\sigma_t \lor \sigma_t^1) \lor I_{\{\alpha(\rho_t) \neq u\}}$$

$$\gamma = \tau I_{\{\alpha(\rho_t) = u\}} + \tau_1 I_{\{\alpha(\rho_t) \neq u\}},$$

we can write

$$I_{\{\sigma_1^1 = u\}} E(Y_{\alpha(\pi)} \mid \mathcal{F}_u) = I_{\{\sigma_1^1 = u\}} E(Y_{\alpha(\pi')} \mid \mathcal{F}_u) \leq I_{\{\sigma_1^1 = u\}} \text{ess sup}_{\pi \in \Pi_{\sigma_1}} E(Y_{\alpha(\pi)} \mid \mathcal{F}_{\alpha(\rho_1)}),$$

since $\alpha(\pi') \in \Pi_{\rho_1}$ and on the set $\{\alpha(\rho_1) = u\}$ we have $\alpha(\pi') = \alpha(\pi)$. Thus we obtain that

$$X_{\sigma_1^1} = \sum_u I_{\{\sigma_1^1 = u\}} \text{ess sup}_{\pi \in \Pi_u} E(Y_{\alpha(\pi)} \mid \mathcal{F}_u)$$

$$\leq \sum_u I_{\{\sigma_1^1 = u\}} \text{ess sup}_{\pi \in \Pi_{\rho_1}} E(Y_{\alpha(\pi)} \mid \mathcal{F}_{\alpha(\rho_1)})$$

and

$$X_{\sigma_1^1} \leq \text{ess sup}_{\pi \in \Pi_{\rho_1}} E(Y_{\alpha(\pi)} \mid \mathcal{F}_{\alpha(\rho_1)})$$

which ends the proof of lemma.

\hfill \square

**Corollary 1.** Let $\sigma_t$, $t = 1, 2, \ldots$ be any predictable strategy. Let us consider the policy $\pi = (\sigma_t, \tau)$, where $\tau = t$ with probability 1. Then

$$X_{\sigma_t} = \text{ess sup}_{\pi \in \Pi_{\sigma_t}} E(Y_{\alpha(\pi)} \mid \mathcal{F}_{\sigma_t}).$$

**Lemma 2.** For any predictable strategy $\sigma_t, t = 0, 1, 2, \ldots$ we have

$$\limsup_{t \to \infty} Y_{\sigma_t} = \limsup_{t \to \infty} X_{\sigma_t}.$$

**Proof.** By the previous lemma and corollary we obtain that $X_{\sigma_t} \geq Y_{\sigma_t}$, which implies that

$$\limsup_{t \to \infty} X_{\sigma_t} \geq \limsup_{t \to \infty} Y_{\sigma_t}.$$

Let $n \geq m$. We get

$$X_{\sigma_n} = \text{ess sup}_{\pi \in \Pi_{\sigma_n}} E(Y_{\alpha(\pi)} \mid \mathcal{F}_{\sigma_n})$$

$$\leq \text{ess sup}_{\pi \in \Pi_{\sigma_m}} E(Y_{\alpha(\pi)} \mid \mathcal{F}_{\sigma_n}) \leq E(\sup_{k \geq m} Y_{\sigma_k} \mid \mathcal{F}_{\sigma_n}).$$

Thus

$$\limsup_{n \to \infty} X_{\sigma_n} \leq \lim_{n \to \infty} E(\sup_{k \geq m} Y_{\sigma_k} \mid \mathcal{F}_{\sigma_n})$$

$$= E(\sup_{k \geq m} Y_{\sigma_k} \mid \bigcup_{n=1}^{\infty} \mathcal{F}_{\sigma_n}) = \sup_{k \geq m} Y_{\sigma_k}.$$
which yields the inequality
\[ \limsup_{n \to \infty} X_{\sigma_n} \leq \limsup_{m \to \infty} Y_{\sigma_k} = \limsup_{n \to \infty} Y_{\sigma_n} \]
and we are done. \( \square \)

We are now in a position to prove Theorems 1 and 2. One can easily check that \( X_{\sigma, \tau_0} \) is a martingale.

**Proof of Theorem 1.** Let us assume that \( \limsup_{t \to \infty} Y_{\sigma_t} = 0 \) on \( \tau^0(\sigma) = \infty \) for a predictable strategy \( \sigma_t, t = 0, 1, 2, \ldots \) By the results from [10] we know that \( X_{\sigma, \tau_0} \) is a martingale. Since \( Y_s \) is uniformly integrable we infer that \( X_{\sigma, \tau_0} \) is convergent almost surely and in \( L_1 \) which yields
\[
\sup_{\pi \in \Pi} EY_{\alpha(\pi)} = EX_0 = \lim_{t \to \infty} EX_{\sigma, \tau_0} = E(\text{lim}_{t \to \infty} \{I_{\{\tau_0 = \infty\}} X_{\sigma_t} + I_{\{\tau_0 < \infty\}} X_{\sigma, \tau_0}) = E\{I_{\{\tau_0 = \infty\}} \limsup_{t \to \infty} Y_{\sigma_t} + E\{I_{\{\tau_0 < \infty\}} X_{\sigma, \tau_0}.
\]
Thus if \( \limsup_{t \to \infty} Y_{\sigma_t} = 0 \) on \( \tau^0 = \infty \) then
\[
\sup_{\pi \in \Pi} EY_{\alpha(\pi)} = EX_0 = EX_{\sigma, \tau_0} = EY_{\sigma, \tau_0}
\]
and \( (\sigma_t, \tau^0(\sigma)) \) is an optimal predictable strategy which ends the proof of Theorem 1. \( \square \)

**Proof of Theorem 2.** Now let us assume that \( \pi' = (\sigma_t', \tau') \) is an optimal predictable strategy. By the assumption we can put \( X_\infty = 0 \). Using the same methods as in [12] we can prove that \( \Pi_{\pi'} \) is directed upwards and
\[
\sup_{\pi \in \Pi_{\pi'}} EY_{\alpha(\pi)} = EX_{\alpha(\pi')},
\]
Thus we have obtained that
\[
EY_{\alpha(\pi')} = \sup_{\pi \in \Pi} EY_{\alpha(\pi)} \geq \sup_{\pi \in \Pi_{\pi'}} EY_{\alpha(\pi)} = EX_{\alpha(\pi')}
\]
Obviously \( Y_{\alpha(\pi')} \leq X_{\alpha(\pi')} \).

Both inequalities \( EY_{\alpha(\pi')} \geq EX_{\alpha(\pi')} \) and \( Y_{\alpha(\pi')} \leq X_{\alpha(\pi')} \) imply that \( Y_{\alpha(\pi')} = X_{\alpha(\pi')} \) almost surely which yields \( \tau' \geq \tau^0(\sigma') \). From Doob’s inequality for supermartingales we obtain that \( EX_{\alpha(\pi')} \geq EX_{\alpha(\pi)}, \) where \( \pi^0 = (\sigma_t', \tau^0) \). Since \( X_{\alpha(\pi')} = Y_{\alpha(\pi')} \) and \( X_{\alpha(\pi)} = Y_{\alpha(\pi^0)} \) we conclude that
\[
EX_{\alpha(\pi')} = EY_{\alpha(\pi')} \geq EX_{\alpha(\pi')} = EX_{\alpha(\pi')} = \sup_{\pi \in \Pi} EY_{\alpha(\pi)}.
\]
Thus \( EX_{\alpha(\pi')} = \sup_{\pi \in \Pi} EY_{\alpha(\pi)} \).

Taking into account the equality
\[
\sup_{\pi \in \Pi} EY_{\alpha(\pi)} = EX_0
\]
\[
= E(\limsup_{t \to \infty} Y_{\sigma_t} I_{\{\tau_0 = \infty\}} + E(X_{\sigma_t'} I_{\{\tau_0 < \infty\}}) = EX_{\sigma_t'},
\]

\[\]
we conclude that \( \limsup_{t \to \infty} Y_{\sigma_t} = 0 \) on the set \( \{ \tau^0 = \infty \} \) which ends the proof of Theorem 2. \( \square \)

There are also other optimization problems connected with observation of random fields. One can use the probability maximizing approach which sometimes seems to be more natural than the presented above the expected value maximization because such approach does not need the integrability assumption for \( Y_t \).

Let us assume that for each \( \omega \in \Omega \) a set \( B(\omega) \subset \mathbb{E} \) is defined and for which \( \{ (\omega, x) : x \in B(\omega) \} \in \mathcal{F} \otimes \mathcal{B}_{\mathbb{R}^n} \), where \( \mathcal{B}_{\mathbb{R}^n} \) is the Borel \( \sigma \)-algebra. In the sequel the set \( B(\omega) \) will be also denoted by \( B \) for shortness. We say that a policy \( \pi^* = (\sigma^*_t, \tau^*) \) is optimal if

\[
P(Y_{\alpha(\pi^*)} \in B, \tau^* < \infty) = \sup_{\pi \in \Pi} P(Y_{\alpha(\pi)} \in B, \tau < \infty).
\]

Let us introduce the random variables \( Z_s = P(Y_s \in B \mid \mathcal{F}_s) \). Similarly as \( Y_\infty \) we define \( Z_\infty = 0, \mathcal{F}_\infty = \bigvee_{s \in S} \mathcal{F}_s \) and

\[
X_s = \text{ess} \sup_{\alpha(\pi) \in \Pi_s} P(Y_{\alpha(\pi)} \in B, \tau < \infty \mid \mathcal{F}_s).
\]

Let

\[
\tau^0(\sigma) = \inf \{ t : X_{\sigma_t} = Z_{\sigma_t} \}.
\]

From Theorems 1 and 2 we infer

**Corollary 2.** If \( \pi^* = (\sigma^*_t, \tau^*) \) is an optimal policy maximizing \( P(Y_{\alpha(\pi)} \in B, \tau < \infty) \) then \( \lim_{t \to \infty} Z_{\sigma_t} = 0 \) on the set \( \{ \tau^0(\sigma^*) = \infty \} \) almost surely. If a predictable policy \( \pi' = (\sigma'_t, \tau') \) is such that \( \lim_{t \to \infty} Z_{\sigma'_t} = 0 \) on the set \( \{ \tau^0(\sigma') = \infty \} \) then the policy \( (\sigma'_t, \tau^0(\sigma')) \) is optimal.

### 3. Optimal stopping of Markov random fields

The proof of Theorem 1 is based on the fact that there exists Snell's envelope for a random field considered at each special case. So, there arises the problem of finding a suitable expression for Snell's envelope. In the stochastic processes case the problem was solved for Markov processes where a nice formula for Snell's envelope was given (see for example [12]). We will follow this way.

Let \( Y_z \) be a random field adapted to a family of \( \sigma \)-algebras \( \{ \mathcal{F}_z \} \). The parameter \( z = (s, t) \) is assumed to belong to \( \mathbb{N} \otimes \mathbb{N} \). We assume that the \( \sigma \)-algebras \( \{ \mathcal{F}_z \} \) satisfy the same property as in Section 2 and additionally fulfill the F4 condition (see Mandelbaum and Vanderbei [10] and Walsh [17]). The random field \( Y_z \) may take values from \( \mathbb{R}^n \).

**Definition 1.** (Koreziloglou H., Lefort P., Mazziotto G. [7]) A random field \( Y_z, z \in \mathbb{N} \otimes \mathbb{N} \), adapted to \( \{ \mathcal{F}_z \} \) is said to have the Markov property if for each \( z = (s, t) \in \mathbb{N} \otimes \mathbb{N}, n \in \mathbb{N}, z_1, z_2, \ldots, z_n \in [0, z]^c \) and any real valued bounded Borel function \( f \) the following equality holds

\[
E(f(Y_{z_1}, Y_{z_2}, \ldots, Y_{z_n}) \mid \mathcal{F}_z) = E(f(Y_{z_1}, Y_{z_2}, \ldots, Y_{z_n}) \mid Y_{z \wedge z_1}, \ldots, Y_{z \wedge z_n}),
\]

\[
z \wedge z_j = (s, t) \wedge (s_j, t_j) = (s \wedge s_j, t \wedge t_j).
\]

It is worth noting that under the assumption that the \( \sigma \)-algebras \( \{ \mathcal{F}_z \} \) satisfy the F4 condition each random field can be transformed into a Markov random field. Namely, we have the following

**Proposition 1.** Let \( \tilde{Y}_z = (Y_{(0,0)}, \ldots, Y_u, \ldots, Y_z), u \leq z, \) and \( \mathcal{G}_z = \bigvee_{u \leq z} \mathcal{F}_u \) then under the assumption that the \( \sigma \)-algebras \( \{ \mathcal{F}_z \} \) satisfy the F4 condition, the random field \( (\tilde{Y}_z, \mathcal{G}_z) \) is a Markov random field.
Remark 1. As the state space of the random field $\tilde{Y}_z$ we can take the space of infinite sequences of elements from $\mathbb{R}^n$ with fixed values on the coordinates except some finite number of the beginning coordinates of these sequences.

By analogy with the stochastic processes case we introduce a class of homogeneous Markov random fields.

**Definition 2.** A Markov random field $Y_z$ is homogeneous if for each $z_1, z_2, h \in \mathbb{N} \otimes \mathbb{N}$, $h \geq (0, 0)$, $y \in \mathbb{E}$ and $B \in \mathbb{R}^n$

$$P(Y_{z_1+h} \in B | Y_{z_1} = y) = P(Y_{z_2+h} \in B | Y_{z_2} = y).$$

As in Dynkin [3] it is easily seen that each Markov random field $Y_z$ can be transformed into a homogeneous Markov random field. Namely, the random field $U_z = (s, Y_s)$ possesses the homogeneity property with an appropriate transition probability. Moreover, taking into account Proposition 1 we conclude that for an arbitrary random field $Y_z$ the random field $U_z = (s, \tilde{Y}_s)$ is a homogeneous Markov random field. It is worth noting that despite the fact that the dimension of the vector $\tilde{Y}_s$ depends on $s$ the appropriate conditional probabilities depend only on the increment $h$ and fixed states. Thus going along the lines of Neveu’s [12] arguments we can prove

**Theorem 3.** Let $Y_z$ be a homogeneous Markov random field with values in a countable state space $\mathbb{E}$. Let

$$P^1(u, t) = P(Y_{(i+1,j)} = t | Y_{(i,j)} = u),$$
$$P^2(u, t) = P(Y_{(i,j+1)} = t | Y_{(i,j)} = u).$$

If $f : \mathbb{E} \to \mathbb{R}_+$ is a function representing the payoff we get deciding to stop after watching $Y_z$, then there exists a function $v$ such that $v(Y_z)$ is Snell’s envelope for the problem of optimal stopping of the random field $Y_z$ (we want to find a predictable strategy maximizing the expectation of the payoff function (cf. [10])). The function $v$ is the limit of the sequence $f_0 = f$, $f_{k+1} = \max \{ f, \mathbb{P}^1 f_k, \mathbb{P}^2 f_k \}$, where

$$\mathbb{P}^1 f(y) = E(f(Y_{(i+1,j)}) | Y_{(i,j)} = y),$$
$$\mathbb{P}^2 f(y) = E(f(Y_{(i,j+1)}) | Y_{(i,j)} = y).$$

The function $v$ fulfills the following equality

$$v(y) = \max \{ f(y), \mathbb{P}^1 v(y), \mathbb{P}^2 v(y) \} \quad \text{for each } y \in \mathbb{E}.$$

*Proof.* By the induction arguments we have that the sequence $f_k$ is increasing and convergent. Let $v(y) = \lim_{k \to \infty} f_k(y)$. Of course, the function $v$ satisfies the equation (3). By the Markov property of the random field $Y_z$ and the definition of the function $v$ we conclude that

$$E(v(Y_z) | \mathcal{F}_z) \leq v(Y_z) \quad \text{for } z \leq u.$$  

Moreover, the sequence $v(Y_z)$ is the smallest supermartingale majorizing $f(Y_z)$. Indeed, let $U_z$ be a supermartingale majorizing $f(Y_z)$ i.e. $U_z \geq f(Y_z) = f_0(Y_z)$. Let us suppose that $U_z \geq f_k(Y_z)$ and $z = (s, t)$. We have then for $u_1 = (s + 1, t)$ and $u_2 = (s, t + 1)$

$$U_z \geq E(U_{u_1} | \mathcal{F}_z) \geq E(f_k(Y_{u_1}) | \mathcal{F}_z) = \mathbb{P}^1 f_k(Y_z),$$
$$U_z \geq E(U_{u_2} | \mathcal{F}_z) \geq E(f_k(Y_{u_2}) | \mathcal{F}_z) = \mathbb{P}^2 f_k(Y_z)$$

and consequently $U_z \geq f_{k+1}(Y_z)$. Thus, by the induction $U_z \geq f_k(Y_z)$ for each $k$ which implies $U_z \geq v(Y_z)$. 


By Mandelbaum and Vanderbei's result [10] we obtain that \( v(Y_z) \) is Snell's envelope and there exists the optimal predictable strategy. □

**Remark 2.** Theorem 3 can be carried over to the case \( z \in \mathbb{N}^k \).

**Remark 3.** By Proposition 1 and Remark 2 we can prove Theorem 3 for an arbitrary random field \((Y_z, \mathcal{F}_z)\) with \( \{\mathcal{F}_z\}_z \) fulfilling the F4-condition. Of course, we have to redefine the payoff function as function of \( z \) and \( Y_z \).

4. The maximizing probability approach to the optimal stopping of several Markov chains. Let \((Y_i^j, \mathcal{F}_i^j, \mathbb{E}^j, P^j), j = 1, 2, \ldots, k\) be a sequence of independent Markov chains and \( \mathcal{F}_i^j = \sigma\{Y_1^j, Y_2^j, \ldots, Y_i^j\} \). The process \( Y_i^j \) takes values in the state space \( \mathbb{E}^j \) and moves according to the transition probability \( P^j \). We can control the Markov chains by allowing one of them to go according to its law and freezing the other ones. Let us define a random field

\[
Y_z = (Y_{t_1}^1, Y_{t_2}^2, \ldots, Y_{t_k}^k) = Y_{(t_1, t_2, \ldots, t_k)}.
\]

The random field \((Y_z, \mathcal{F}_z)\), where \( \mathcal{F}_z = \sigma(\mathcal{F}_1^1, \mathcal{F}_2^2, \ldots, \mathcal{F}_k^k) \), is a Markov random field in the sense of Definition 1. Our aim is to apply the maximizing probability approach to the optimal stopping of the random field \( Y_z \). The optimality is meant in the sense of (1). Let \( Z_s = P(Y_s \in B(\mathcal{F}_s)) = f(s, Y_s) = f(U_s) \), where \( s = (t_1, t_2, \ldots, t_k) \) and \( Y_s \) is defined in Proposition 1. By this proposition and Remark 2 the process \( U_z = (s, Y_s) \) is a homogeneous Markov random field. By Corollary 1 and Theorem 3 (see also arguments in [10]) the optimal strategy can be described in the following way. Let

\[
\mathbb{P}^j f(u) = E(f(U_{(t_1, t_2, \ldots, t_j-1, t_j+1, t_{j+1}, \ldots, t_k)})\mid U_{(t_1, t_2, \ldots, t_k)} = u),
\]

\( j = 1, 2, \ldots, k \) and let \( \Gamma, \Gamma^j \) be a partition of the range of the process \( U \) such that \( \Gamma^j \subset \{v = \mathbb{P}^j v\} \cup \{v = 0\}, \Gamma \subset \{v = f\} \), where the function \( v \) fulfils the equation

\[
v(u) = \max\{f(u), \mathbb{P}^1 v(u), \mathbb{P}^2 v(u), \ldots, \mathbb{P}^k v(u)\}.
\]

We can put \( \sigma_{t+1} = \sigma_t + e_j, \) where \( (\sigma_t, Y_{\sigma_t}) \in \Gamma^j \) \( e_j \) is the unit vector with 1 in the \( j \)-th coordinate and 0 elsewhere. We stop when \( (\sigma_t, Y_{\sigma_t}) \) hits \( \Gamma \) at the first time. The theorems formulated in the previous sections lead us to the conclusion that such a strategy is optimal if \( \tau^* = \inf\{t : (\sigma_t, Y_{\sigma_t}) \in \Gamma^j\} < \infty \) almost surely.

**Example 1.** The example concerns the problem of optimal allocation of different treatments (cf. [5]). Assume that a physician has two different treatments at his disposal which can be applied sequentially and can be interchanged during a cure. Assume also that application of either of these treatments does not change the current patient's chances of recovering after applying the other treatment. Another words, we do not observe the presence of an immediate synergistic effect. Mathematically, taking into account the simplest case, the procedure can be described by two independent Markov chains \( Y_{t_1}^1, Y_{t_2}^2, t_1 = 0, 1, \ldots, N, t_2 = 0, 1, \ldots, N \) with state space \( \mathbb{E} = \{0, 1\} \) and \( Y_0^1 = Y_0^2 = 0 \) almost surely. \( Y_{t_i}^i, i = 1, 2, \) represents the patient's state of health after \( t_i \)-times application of \( i \)-th treatment. Thus we have a random field \( Y_{(t_1, t_2)} = (Y_{t_1}^1, Y_{t_2}^2) \) which describes the patient's therapy. Having observed the random field \( Y_{(t_1, t_2)} \), the therapy can be stopped and after some fixed period of time the patient is again examined for his state of health. The result of the examination is described by the random variable \( \Theta \). Taking on values 0 or 1. We assume that we know the following conditional distribution of the random variable \( \Theta \) (the distribution is conditioned by the whole history of a therapy)

\[
P(\Theta = m \mid Y_u^1 = k_u, Y_w^2 = l_w, u \leq t_1, w \leq t_2) = Q(m, k_u, l_w, t_1, t_2, u \leq t_1, w \leq t_2).
\]
Our aim is to find a predictable policy $\pi^* = (\sigma^*_r, \tau^*) = ((\delta_1, r, \delta_2, r), \tau^*)$ such that

$$P(Y_{\delta_1, r}^1 = Y_{\delta_2, r}^2 = \Theta = 1) = \sup_{(\sigma, \tau) = ((\delta_1, r, \delta_2, r), \tau)} P(Y_{\delta_1, r}^1 = Y_{\delta_2, r}^2 = \Theta = 1).$$

Let

$$Z_s = Z_{(t_1, t_2)} = P(Y_{t_1}^1 = Y_{t_2}^2 = \Theta = 1 | \mathcal{F}_s) = \begin{cases} Q(1, k_u, l_w, t_1, t_2, u < t_1, w < t_2) & \text{if } Y_{t_1}^1 = Y_{t_2}^2 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus $Z_s = f(s, \tilde{Y}_s)$. According to the previous considerations our next step is to find the function $v$—the smallest multi-excessive majorant satisfying the equation

$$v(u) = \max\{f(u), \mathbb{P}^1v(u), \mathbb{P}^2v(u)\}.$$

To solve this equation we can use the approximation proved in Theorem 3. Having this equation solved we can divide the state space of the process $U_s = (s, \tilde{Y}_s)$ onto the sets $\Gamma$, $\Gamma^1$, $\Gamma^2$ and define the optimal predictable strategy in the same way as it is done at the beginning of the section.

The model can be generalized to the case when a synergistic effect of investigated treatment is present. We can also apply the model to an experimental psychology and system theory (see Tanaka [16]).

5. Some problem of optimal selection. Let us consider the problem of a person who wants to buy an asset. He admits possibility to do that in one of $k$ towns during $N$ days. Every day each town puts some asset out for sale. At the first day our decision maker is in the first town and he can decide if he buys the asset or if he rejects the proposition. If he rejects he has to decide if he wants to stay at the same town or to go to the second one. He cannot return to the previous proposition and to the towns visited previously. Leaving the town he gets possibility to have information about assets presented at this town. Arriving to new town he gets information about the previous and actual propositions at the town. The assets are different and the aim of the decision maker is to accept the best possible.

**Table 1.** The strategies for relative rank 1

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**Table 2.** The strategies for relative rank > 1

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Another problem of similar structure is following. There are $k$ work lines. Each line produce goods of the same kinds but different in quality. During $N$ days we have to choose the best one. The conditions of the selection process are following. At each day we can analyze and decide about the product of one line and additionally we get the information about products from lines investigated previously. So, one can observe $N\cdot k$ different objects which can be numbered according to their rank. The best has rank 1. All arrangements of products are equally likely. The expert can make one of the following decisions: to investigate next object at the same line or to change the current line and to investigate the object on the next line or to stop the investigation and to decide that the observed object is the best one. We assume that the expert resigning from the $i$-th object cannot come back to that one and moreover he is not able to come back to earlier objects in the next lines. Nevertheless the expert investigating the $i$-th object at $j$-th line is able to gather the information about all earlier objects which came to the previous lines. For example, the objects refused to be the best can be investigated irrespective of expert’s opinion and sent for other purposes but the result of the investigation is sent to the expert. The aim of the expert is to recognize the best object with maximal probability.

We can formulate the above problems in the following way. By $v(r,s)$ we denote the absolute rank of the $s$-th object coming to the $r$-th line. The random variables $v(r,s)$, $1 \leq r \leq k$, $1 \leq s \leq N$, are defined on the set of all permutation of the set $\{1,2,\ldots,kN\}$. The best object has the rank 1. Let $Y(r,s) = \text{card}\{(i,j) : 1 \leq i \leq s, 1 \leq j \leq N, v(i,j) \leq v(r,s)\}$. The random variable $Y_z = Y(r,s)$ is called a relative rank of the $s$-th object at the $r$-th line. We assume that we are only able to observe the random variables $Y_z$. Thus we can define the following probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_z\}, P)$, where $\Omega$ is the set of all permutations of $kN$-element set, $\mathcal{F}$ - the set of all subsets of $\Omega$, $\mathcal{F}_z = \sigma\{Y_u, u = (i,j) \leq z = (r,s)\}$ and $P$ - the probability defined on $\mathcal{F}$ under the assumption that all permutations are equally probable. Our aim is to find a predictable strategy $\pi^* = (\sigma^*, \tau^*)$ which maximizes $P(v_{\sigma(\tau)} = 1)$. The restriction to the class of predictable strategies is motivated by the conditions of choice presented above. Let $g(i)$ be the payoff function defined as follows.

$$g(i) = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{otherwise}. \end{cases}$$
For any predictable strategy \( \pi = (\sigma_t, \tau) \) we can write

\[
P(v_{\alpha(\pi)} = 1) = E g(v_{\alpha(\pi)}) = \sum_z \int_{\{\alpha(\pi) = z\}} g(v_z) dP = \sum_z \int_{\{\alpha(\pi) = z\}} P(v_z = 1 | \mathcal{F}_z) dP = Ef(\alpha(\pi), Y_{\alpha(\pi)}),
\]

where

\[
f(z, y) = P(v_z = 1 | Y_z = y) = \begin{cases} \frac{rs}{kN} & \text{if } y = 1 \\ 0 & \text{otherwise,} \end{cases}
\]

\( z = (r, s), 1 \leq y \leq rs \). By the same argument as in the one-parameter case (see e.g. [15]) we conclude that the random variables \( Y_z \) are independent and for \( z = (r, s) \) we have \( P(Y_z = y) = 1/(rs) \). Thus the random field \( Y_z \) is a Markov random field. Moreover, the random filed \( U_z = (s, Y_s) \) is homogeneous Markov random filed with the state space \( \mathbb{E} = \{((i, j), y) : 1 \leq i \leq k, 1 \leq j \leq N, 1 \leq y \leq kN\} \) and an appropriate transition probabilities. Using this fact we can solve the problem of optimal stopping of the homogeneous random field \( U_z \) with the payoff function \( f(U_z) = f(s, Y_s) \) defined before. According to Theorem 3 there exists a multi-excessive function \( v \) which is the solution of the equation

\[
v((i, j), y) = \max \{f((i, j), y), \mathbb{P}^1 v((i, j), y), \mathbb{P}^2 v((i, j), y)\}.
\]

Since the equation gives a recursive formula for \( v \) it can be easily solved. Namely, we get

\[
v((i, j), y) = \max \{f((i, j), y), \ \frac{1}{(i+1)j} \sum_{y=1}^{(i+1)j} v((i+1, j), y),
\]

\[
\frac{1}{i(j+1)} \sum_{y=1}^{i(j+1)} v((i, j+1), y)\}.
\]

Snell’s envelope is equal to \( v(U_z) = v((s, Y_s)) \). The optimal predictable strategy can be described similarly as in Mandelboun and Vanderbei [10]. Let \( \Gamma, \Gamma^1, \Gamma^2 \) be a partition of the state-space \( \mathbb{E} \). Define \( \Gamma = \{v = f\}, \Gamma^1 = \{v = \mathbb{P}^1 v \cup \{v = 0\}, \Gamma^2 = \{v = \mathbb{P}^2 v \cup \{v = 0\}, \sigma_{t+1} = \sigma_t + e_k \) where \( e_k = (1, 0) \) if \( U_{\sigma_t} \in \Gamma^1 \) and \( e_k = (0, 1) \) if \( U_{\sigma_t} \in \Gamma^2 \). We stop at the first moment \( t \) such that \( U_{\sigma_t} \in \Gamma \). The policy defined above is optimal and

\[
v_0 = \sup_{\pi \in \Pi} P(v_{\alpha(\pi)} = 1) = \sup_{\pi \in \Pi} Ef(U_{\alpha(\pi)})
\]

\[
= v(U_{(1,1)}) = v((1, 1), 1).
\]

Let us consider a numerical example for \( N = 10 \) and \( k = 4 \). The strategies are presented in Table 1 and 2 (s-stop, fr-e_k = (1, 0), fu-e_k = (0, 1)). In Tables 3 and 4 contains \( v((i, j), y) \). One can ask how the maximal probability of recognizing the best object is changing with \( N \) when \( k \) is fixed. Few numerical values are given in Table 5. It is also interesting problem the behaviour of \( v_N \) when \( N \to \infty \). At the first let us mention that there exists a predicable strategy for which the probability of finding the best object is greater than some positive number for each \( N \). Let \( N \) be even. Let us consider the following strategy. We go to the \( k \)-th line omitting the first \( N/2 \) objects in all lines and we stop at the first object which is the best of all objects previously investigated. We will be certainly successful if the second
object, according to his quality, will be among the refused ones and the best object will be found among the \(N/2\) observed objects. The probability \(p_N\) of such event is equal to

\[
\frac{(N/2)k}{Nk} \cdot \frac{N/2}{(N-1)k} > \frac{1}{4k},
\]

thus \(v_N > \frac{1}{4k}\).

### Table 5. The values of the problem for different \(N\) and \(k\)

<table>
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<td>.1925</td>
<td>.1575</td>
<td>.1337</td>
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Now, we derive the limit of \(v_N\) when \(N \to \infty\). Since \(k\) is fixed we have from (4) that after some steps in optimal procedure the decision process reaches the line \(k\) and it stays there to the end. All \(Nk\) points one can consider linearly ordered. We have

\[
v((r,k),1) = \sum_{i=r+1}^{N} \frac{r}{i(i-1)} \left[ \frac{1}{k} \frac{i}{N} + \frac{k-1}{k} v((i,k),1) \right].
\]

When \(N \to \infty\) such that \(\frac{N}{k} \to x\) we get

\[
v(x) = \lim_{N \to \infty} v((r,k),1)
\]

\[
= \int_{x}^{1} \frac{1}{t^{2}} \frac{1}{k} \left[ k - \frac{k-1}{k} v(t) \right] dt.
\]

Hence \(v(x) = \frac{x}{k-1} (x^{\frac{1}{k}-1})\). When the decision process is on \(k\)-th line the optimal strategy is to stop on relatively first observation at moment \(r\) if only \(v((r,k),1) < \frac{N}{k}\). In limiting case we have to accept object relatively best at \((r,k)\) when \(r \geq [x_{\alpha} N]\), where \(x_{\alpha} = k^{\frac{1}{k} - k}\). Since \(k\) is fixed we have for \(k \neq 1\) that \(v(0) = \lim_{N \to \infty} v_N = k^{\frac{1}{k}-k}\). For \(k = 1\) the limiting value is \(e^{-1}\) (cf. [14], [15], [4]).

### 6. Acknowledgment

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OPPORTINAL STOPPING OF RANDOM FIELDS

REFERENCES


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