Markov Stopping Games with Random Priority

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Abstract: In the paper a construction of Nash equilibria for a random priority finite horizon
two-person non-zero sum game with stopping of Markov process is given. The method is used to
solve the two-person non-zero-sum game version of the secretary problem. Each player can choose
only one applicant. If both players would like to select the same one, then the lottery chooses the
player. The aim of the players is to choose the best candidate. An analysis of the solutions for
different lotteries is given. Some lotteries admit equilibria with equal Nash values for the players.

Key Words and Phrases: stopping time, Markov process, non-zero sum game, random priority,
randomize stopping time.

1 Introduction

In the paper a construction of Nash equilibria for a random priority finite horizon
two-person non-zero sum game with stopping of Markov process is given. Let \((X_n, \mathcal{F}_n, P_x)\)\(n=0\) be a homogeneous Markov process defined on a
probability space \((\Omega, \mathcal{F}, P)\) with a state space \((\mathcal{E}, \mathcal{A})\). At each moment \(n = 1, 2,\)
\(\ldots, N\) the decision makers (henceforth called Player 1 and Player 2) are able to
observe the Markov chain sequentially. Each player has his utility function
\(g_i: \mathcal{E} \rightarrow \mathbb{R}, i = 1, 2,\) and at each moment \(n\) each decides separately if he accepts
or rejects the realization \(x_n\) of \(X_n\). We admit \(g_i\) measurable and bounded. If it
happens that both players have selected the same moment \(n\) to accept \(x_n\), then
a lottery decides which player gets the right (priority) of the acceptance.
According to the lottery, at moment \(\tau\), Player 1 is chosen with a probability \(z_\tau\),
while Player 2 with \(\beta_\tau = 1 - z_\tau\). The player which has been rejected by the
lottery may select any other realization \(x_n\) in the later moments \(n, \tau < n \leq N\).
Once accepted realization cannot be rejected, once rejected cannot be

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reconsidered. If a player has not chosen any realization of Markov process he
gets \( g^* = \inf_{x \in :} g_d(x) \). The aim of each player is to choose a realization which
maximizes his expected utility. In fact, the problem will be formulated as a two
person non-zero sum game with the concept of the Nash equilibrium as the
solution. The problem with permanent priority for Player 1 (i.e. \( x_n = 1, n = 1, 2, \ldots \))
has been solved by Ferenstein (1992). This game is also strictly connected
with optimal stopping of stochastic processes. The ideas of Kuhn (1953) and
Rieder (1979) as well as Yasuda (1983) and Ohtsubo (1987) will be adopted to this
random priority game model. Based on this approach we deal with non-
cooperative two-person time sequential non-zero-sum game version of the best
choice problem (the secretary problem). We focus our attention on the general-
ization of a game model of the problem considered by Fushimi (1981). For the
original secretary problem and its extension the reader is referred to Gilbert &
Mosteller (1966), Freeman (1983) or Rose (1982). We recall the best choice
problem for completeness here.

An employer is to view a group of \( N \) applicants for a vacancy sequentially.
Each of the applicants has some characteristics unknown to the employer. We
assume that all permutations of them are equally likely. Let \( Z_k \) denote the
absolute rank of the applicant with the characteristics \( X_k \), i.e. \( Z_k = \text{card}\{1 \leq i \leq N: X_i \leq X_k\} \), (card(A) denotes cardinality of the set A). The decisions of the
employer at each time \( n \) are based on the relative rank \( Y_1, Y_2, \ldots, Y_n \) of the
applicants, where \( Y_k = \text{card}\{1 \leq i \leq k: X_i \leq X_k\} \). The objective of the employer
is to maximize the probability of choosing the applicant with absolute rank 1.

Many authors have considered games inspired by this problem. Let us men-
tion the papers by Enns and Ferenstein (1987), Fushimi (1981), Majumdar
(1986), Sakaguchi (1989, 1991), Ravindran and Szajowski (1992) and Szajowski
(1992) where non-zero sum versions of the games have been investigated. A
review of these problems one can find in Ravindran and Szajowski (1992). In
non-cooperative non-zero sum games one of possible definitions of solution is
Nash equilibrium. This approach gives very often many different solutions (pairs
of strategies for the players) with various gains for the players. Many authors
neglect this problem and they give construction of some solution. However,
players do not communicate with each other. When there are many solutions,
then there is a question which one should be used by players. It could happen
that they choose strategies from different equilibrium points. In such a case
neither of them obtains the expected gain but less (if the aim of the players is to
maximize their return). That is why it is important to have knowledge about all
possible solutions of the game. Investigation of alternative solutions is also an
interesting theoretical problem. Consideration in this direction for the matrix
games can be found, for instance, in Moulin (1986). A tip from two person
non-zero sum generalized secretary problem with fixed priority has been given
A very interesting illustration of the problem in stopping games are the models of
two person best choice problems considered by Fushimi (1981). One of them is
generalized in this paper and can be described as follows.
Two companies (Player 1 and Player 2) interview a sequence of applicants one by one (as in the best choice problem which has been recalled above) every morning independently of the other company, and the results of the interviews are communicated to the applicant in the afternoon. If only one of the companies decides to accept the applicant, she agrees to this offer at once, the other company is informed of this fact and continues the interviewing process. If, on the other hand, both companies decide to accept the applicant, she selects one of them with equal probabilities and the other company can continue interviewing and employ another applicant. In Fushimi (1981) the threshold strategies for the players were admitted. It was shown that equilibrium strategies for players in the model are different. One of the players should behave more hastily than in the original secretary problem and he should start solicitation at \( .2865 \) for the limiting version of the problem. There are two Nash equilibria in the considered set of strategies for this game with values \((.2865, .2963)\) and \((.2963, .2865)\), respectively.

In this paper the generalization of the \( x \) problem has been considered. It is assumed that if both companies want to accept the same applicant, Player 1 is selected with fixed probability \( x \), Player 2 with probability \( 1 - x \), \( x \in [0, 1] \), and the player who has not been chosen continues interviewing and employs another applicant. Also a more general set of strategies is admitted. This particular game problem is presented as interesting per se. The mathematical model of the above formulated problem will be presented and equilibria for each \( x \) will be derived in Section 3. This section points out interesting properties of some solutions. The rigorous definitions of strategies and other problems which appeared in this game version of the secretary problem are the subject of Section 2.

2 The Game with Random Priority

In the problem of optimal stopping the basic class of strategies \( \mathcal{F}^N \) are Markov times with respect to \( \sigma \)-fields \( \{\mathcal{F}_n\}_{n=1}^N \). We admit that \( P(\tau \leq N) < 1 \) for some \( \tau \in \mathcal{F}^N \). This class of strategies is not sufficient in the stopping game (see Yasuda (1985)). So we consider a class of randomized stopping times. It is assumed that the probability space is rich enough to admit the following constructions.

Definition 1: (see Yasuda (1985)) A strategy for each player is a random sequence \( p = (p_n) \in \mathcal{P}^N \) or \( q = (q_n) \in \mathcal{P}^N \) such that, for each \( n \), (i) \( p_n, q_n \) are adapted to \( \mathcal{F}_n \); (ii) \( 0 \leq p_n, q_n \leq 1 \) a.s.. If each random variables equals either 0 or 1 we call it a pure strategy.
Let $A_1, A_2, \ldots, A_N$ and $B_1, B_2, \ldots, B_N$ be i.i.d. r.v. of the uniform distribution on $[0,1]$ and independent of Markov process $(X_n, \mathcal{F}_n, P_{x_n})$. Let $\mathcal{F}_n$ be the $\sigma$-field generated by $\mathcal{F}_n, \{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$. A random Markov time $\lambda(p)$ for strategy $p = (p_n) \in \mathcal{P}_N$ and $\mu(q)$ for strategy $q = (q_n) \in \mathcal{M}_N$ are defined by $\lambda(p) = \inf\{N \geq n \geq 1: A_n \leq p_n\}$ and $\mu(q) = \inf\{N \geq n \geq 1: B_n \leq q_n\}$, respectively. We denote by $\mathcal{A}_N$ and $\mathcal{M}_N$ the sets of all randomized strategies of Player 1 and Player 2. Clearly, if each $p_n$ is either zero or one, then the strategy is pure and $\lambda(p)$ is in fact an $\{\mathcal{F}_n\}$-Markov time. In particular an $\{\mathcal{F}_n\}$-Markov time $\lambda$ corresponds to the strategy $p = (p_n)$ with $p_n = \mathbb{I}_{\lambda = n}$, where $\mathbb{I}_A$ is an indicator function of the set $A$.

In the considered game the class of randomized Markov times is not adequate because they allow that both players can stop at the same time. The random assignment of the priority to the player requires to consider the modified strategies. Denote $\mathcal{F}_n^N = \{\tau \in \mathcal{F}_N: \tau \geq k\}$. One can define the set of strategies $\mathcal{A}_N = \{p, \{\sigma_n\} : p \in \mathcal{P}_N, \{\sigma_n\} \in \mathcal{F}_n^N\}$ for every $n$ and let $\mathcal{M}_N = \{q, \{\sigma_n\} : q \in \mathcal{M}_N, \{\sigma_n\} \in \mathcal{F}_n^N\}$ for every $n$. For Player 1 and Player 2, respectively.

Let $\xi_1, \xi_2, \ldots$ be i.i.d. uniformly distributed on $[0,1]$ and independent of $\chi_n$ and the lottery is given by $\tilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N)$. Denote $\mathcal{N}_n = \sigma\{\mathcal{H}_n, \xi_1, \xi_2, \ldots, \xi_n\}$ and let $\mathcal{F}_n$ be the set of Markov times with respect to $\mathcal{H}_n$. For every pair $(s, t)$ such that $s \in \mathcal{A}_N$, $t \in \mathcal{M}_N$ we define $
abla_1(s, t) = \lambda(p) + \lambda(p) - \mu(q) + \mu(q)$ and $
abla_2(s, t) = \mu(q) + \mu(q)$ for every $s \in \mathcal{A}_N$ and $t \in \mathcal{M}_N$.

**Definition 2:** The Markov times $\tau_1(s, t)$ and $\tau_2(s, t)$ are selection times of Player 1 and Player 2 when they use strategies $s \in \mathcal{A}$ and $t \in \mathcal{M}$, respectively, and the lottery is $\tilde{\alpha}$.

For each $(s, t) \in \mathcal{A}_N \times \mathcal{M}_N$ and given $\tilde{\alpha}$ the payoff function for the $i$-th player is defined as $f_i(s, t) = g_i(X_{t,x(s,t)})$. Let $\tilde{R}_i(s, t) = E_{x} f_i(s, t)$ be the expected gain of $i$-th player if the players use $(s, t)$. We have defined the game in normal form $(\tilde{A}_N, \tilde{M}_N, \tilde{R}_1, \tilde{R}_2)$. This random priority game will be denoted $\mathcal{G}_{rp}$.

**Definition 3:** A pair $(s^*, t^*)$ of strategies such that $s^* \in \mathcal{A}_N$ and $t^* \in \mathcal{M}_N$ is called a Nash equilibrium in $\mathcal{G}_{rp}$ if for all $x \in \mathbb{E}$

$$v_1(x) = \tilde{R}_1(s, t^*) \geq \tilde{R}_1(s, t) \quad \text{for every } s \in \mathcal{A}_N,$$

$$v_2(x) = \tilde{R}_2(s^*, t) \geq \tilde{R}_2(s, t) \quad \text{for every } t \in \mathcal{M}_N.$$

The pair $(v_1(x), v_2(x))$ will be called the Nash value.
Denote $h_i(n, X_n) = \text{ess sup}_{x \in F_n} E_{X_n} g_i(X)$ and $\sigma^*$ a stopping time such that $h_i(0, x) = E_{X_n} g_i(X)$ for every $x \in \mathbb{R}^i$, $i = 1, 2$. Let $\Gamma^i = \{x \in \mathbb{R}: h_i(n, x) = g_i(x)\}$. We have $\sigma^* = \text{inf} \{n: X_n \in \Gamma^i\}$ (cf. Shiryaev (1978)). Denote $\sigma^* = \text{inf} \{n > k: X_n \in \Gamma_n\}$. Taking into account the above definition of $\mathcal{Y}_p$, one can conclude that the Nash values of this game are the same as in the auxiliary game $\mathcal{G}_{wp}$ with the sets of strategies of the players $\mathcal{P}^N$, $\mathcal{Q}^N$ and payoff functions (cf. Yasuda (1985))

$$\varphi_1(p, q) = g_1(X_{\lambda(p)}) \mathbb{P}_{\lambda(p) < \mu(q)} + \tilde{h}_1(\mu(q), X_{\mu(q)}) \mathbb{P}_{\lambda(p) > \mu(q)}$$

$$+ [g_1(X_{\lambda(p)}) \mathbb{P}_{\lambda(p) < \mu(q)} + \tilde{h}_1(\lambda(p), X_{\lambda(p)}) (1 - \mathbb{P}_{\lambda(p)})] \mathbb{P}_{\lambda(p) = \mu(q)}$$

$$\varphi_2(p, q) = g_2(X_{\mu(q)}) \mathbb{P}_{\mu(q) < \lambda(p)} + \tilde{h}_2(\lambda(p), X_{\lambda(p)}) \mathbb{P}_{\mu(q) = \lambda(p)}$$

$$+ [g_2(X_{\lambda(p)}) (1 - \mathbb{P}_{\lambda(p)}) + \tilde{h}_2(\lambda(p), X_{\lambda(p)}) \mathbb{P}_{\lambda(p) = \mu(p)}] \mathbb{P}_{\lambda(p) = \mu(p)}$$

for each $p \in \mathcal{P}$, $q \in \mathcal{Q}$, where $\tilde{h}_i(n, X_n) = \text{ess sup}_{x \in F_{n+1}} E_{X_n} g_i(X) = E_{X_n} h_i(n + 1, X_{n+1})$. Denote $R_x(p, q) = E_{X_n} g_i(p, q)$ for every $x \in \mathbb{R}^i$, $i = 1, 2$.

Let $\mathcal{P}_n = \{p = (p_n) \in \mathcal{P}: p_1 = \cdots = p_{n-1} = 0, p_n = 1\}$ and $\mathcal{Q}_n = \{q = (q_n) \in \mathcal{Q}: q_1 = \cdots = q_{n-1} = 0, q_n = 1\}$. We will use the following convention: if $p \in \mathcal{P}_n$ then $(p_n, p)$ is the strategy belonging to $\mathcal{P}_n$ in which the $n$-th coordinate is changed to $p_n$.

**Definition 4:** A pair $(p^*, q^*) \in \mathcal{P}_n \times \mathcal{Q}_n$ is called an equilibrium point of $\mathcal{G}_{wp}$ at $n$ if

$$v_1(n, X_n) = E_{X_n} \varphi_1(p^*, q^*) \geq E_{X_n} \varphi_1(p, q^*) \text{ for every } p \in \mathcal{P}_n, \mathbb{P}_X\text{-a.s.}$$

$$v_2(n, X_n) = E_{X_n} \varphi_2(p^*, q^*) \geq E_{X_n} \varphi_2(p^*, q) \text{ for every } q \in \mathcal{Q}_n, \mathbb{P}_X\text{-a.s.}$$

A Nash equilibrium point at $n = 0$ is a solution of $\mathcal{G}_{wp}$. The pair $(v_1(0, X), v_2(0, X))$ of values is a Nash value corresponding to $(p^*, q^*) \in \mathcal{P}_n \times \mathcal{Q}_n$.

**Theorem 1:** There exists a Nash equilibrium $(p^*, q^*) \in \mathcal{P}_n \times \mathcal{Q}_n$ in the game $\mathcal{G}_{wp}$. The Nash value and an equilibrium point can be calculated recursively.

**Proof:** At moment $N$ the players play the following bimatrix game

$$\begin{pmatrix}
(g_1(N, X_N), g_2(N, X_N)) & (g_1(X_N), g_2^*) \\
(g_1^*, g_2(N, X_N)) & (g_1^*, g_2^*)
\end{pmatrix}$$
where \( \tilde{g}_1(n, x) = \alpha_n g_1(x) + (1 - \alpha_n) \tilde{h}_1(n, x) \) and \( \tilde{g}_2(n, x) = (1 - \alpha_n) g_2(x) + \alpha_n \tilde{h}_2(n, x) \). This game always has an equilibrium in pure or randomized strategies on \( \{\omega: X_n = x\} \) for every \( x \in \mathcal{E} \). We denote a Nash equilibrium in \( \mathcal{P}_N \times \mathcal{Q}_N \) by \((p^*_n, q^*_n)\) and the corresponding Nash value by \((v_1(N, x), v_2(N, x))\). Let us assume that an equilibrium \((p^*, q^*) \in \mathcal{P}_{n+1} \times \mathcal{Q}_{n+1}\) has been constructed and \((v_1(n+1, x), v_2(n+1, x))\) is the Nash value corresponding to this strategy on \( \{\omega: X_n = x\} \). We consider the following bimatrix game

\[
\begin{pmatrix}
(\tilde{g}_1(n, X_n), \tilde{g}_2(n, X_n)) & (g_1(X_n), \tilde{h}_2(n, X_n)) \\
(\tilde{h}_1(n, X_n), g_2(X_n)) & (\tilde{v}_1(n, X_n), \tilde{v}_2(n, X_n))
\end{pmatrix}
\]

where \( \tilde{v}(n, x) \) is such that \( \tilde{v}(n, X_n) = \mathbb{E}_{X_n} v_j(n + 1, X_{n+1}), j = 1, 2 \). On the set \( \{\omega: X_n = x\} \) there is at least one equilibrium point in pure or randomized strategies in this bimatrix game. By measurability of \( g_i(x) \) there exists \((p^*_n, q^*_n)\) such that \( p^*_n, q^*_n \in \mathcal{F}_n \) and \((p^*_n, q^*_n)\) is a Nash equilibrium in the above bimatrix game. We are now in a position to show that \(((p^*_n, p^*), (q^*_n, q^*))\) is an equilibrium of \( \mathcal{G}_{wp} \) in \( \mathcal{P}_N \times \mathcal{Q}_N \). Let \((p_n, p) \in \mathcal{P}_n\), where \( p \in \mathcal{P}_{n+1} \). By properties of conditional expectation and induction assumption we have \( \mathbb{P}_x \)-a.s.

\[
\mathbb{E}_{X_n} \varphi_1((p_n, p), (q^*_n, q^*)) = p_n q^*_n \tilde{g}_1(n, X_n) + p_n (1 - q^*_n) g_1(X_n)
+
(1 - p_n) q^*_n \tilde{h}_1(n, X_n)
+
(1 - p_n) (1 - q^*_n) \mathbb{E}_{X_n} \mathbb{E}_{X_{n+1}, \varphi_1(p, q^*)}
\leq p_n q^*_n \tilde{g}_1(n, X_n) + p_n (1 - q^*_n) g_1(X_n)
+
(1 - p_n) q^*_n \tilde{h}_1(n, X_n)
+
(1 - p_n) (1 - q^*_n) \mathbb{E}_{X_n} v_1(n + 1, X_{n+1})
= v_1(n, X_n)
\]

for each \( x \in \mathcal{E} \). The same is valid for Player 2. This proves the theorem.

The solution of the game \( \mathcal{G}_{wp} \) can be constructed based on the solution \((p^*, q^*)\) of the corresponding game \( \mathcal{G}_{wp} \).

**Theorem 2:** Game \( \mathcal{G}_{wp} \) has a solution. The pair \((s^*, t^*)\), where \( s^* = (p^*, \sigma_{s1}^*) \in \mathcal{A}_N \) and \( t^* = (q^*, \sigma_{t2}^*) \in \mathcal{M}_N \), is an equilibrium point. The value of the game is \((v_1(0, x), v_2(0, x))\).
Proof: Denote for \((s^*, t^*)\) the selection times \(\tau_1^* = \tau_1(s^*, t^*)\) and \(\mu_\pi^* = \tau_2(s^*, t^*)\) of Player 1 and Player 2, respectively. Since process \((X_n, \mathcal{F}_n, P_X)_{n=0}^\infty\) is Markov process and by properties of the conditional expectation, taking into account the definition of \((\lambda^*_\pi, \mu^*_\pi)\), we have

\[
\widetilde{R}_1(x, s^*, t^*) = E_x g_1(X_{\lambda^*_\pi})
\]

\[
= E_x g_1(X_{\lambda(p^*)})1_{\{\lambda(p^*) < \mu(q^*)\}} + E_x g_1(X_{\sigma^*_n})1_{\{\lambda(p^*) > \mu(q^*)\}}
\]

\[
+ E_x [g_1(X_{\lambda(p^*)})1_A + g_1(X_{\sigma^*_n})1_{A^c}]1_{\{\lambda(p^*) = \mu(q^*)\}}
\]

\[
= E_x g_1(X_{\lambda(p^*)})1_{\{\lambda(p^*) < \mu(q^*)\}}
\]

\[
+ E_x E[g_1(X_{\sigma^*_n})1_{\{\lambda(p^*) > \mu(q^*)\}} | \mathcal{F}_{\mu(q^*)}]
\]

\[
+ E_x g_1(X_{\lambda(p^*)})1_A1_{\{\lambda(p^*) = \mu(q^*)\}}
\]

\[
+ E_x E[g_1(X_{\sigma^*_n})1_{A^c}1_{\{\lambda(p^*) = \mu(q^*)\}} | \mathcal{F}_{\mu(q^*)}]
\]

\[
= E_x g_1(X_{\lambda(p^*)})1_{\{\lambda(p^*) < \mu(q^*)\}} + E_x E_{\lambda(q^*)} g_1(X_{\sigma^*_n})1_{\{\lambda(p^*) > \mu(q^*)\}}
\]

\[
+ E_x [g_1(X_{\lambda(p^*)})1_A + [E_{\lambda(q^*)} g_1(X_{\sigma^*_n})]1_{A^c}]1_{\{\lambda(p^*) = \mu(q^*)\}}
\]

\[
= E_x g_1(X_{\lambda(p^*)})1_{\{\lambda(p^*) < \mu(q^*)\}} + E_x h_1(\lambda(p^*), X_{\lambda(p^*)})1_{\{\lambda(p^*) > \mu(q^*)\}}
\]

\[
+ E_x [g_1(X_{\lambda(p^*)})1_A + h_1(\lambda(p^*), X_{\lambda(p^*)})1_{A^c}]1_{\{\lambda(p^*) = \mu(q^*)\}}
\]

\[
= E_x \varphi_1(p^*, q^*)
\]

\[
= R_1(x, p^*, q^*)
\]

where \(A = \{\omega: \tilde{\xi}_{\lambda(p^*)} \leq \tilde{\xi}_{\lambda(p^*)}\}\) and \(A^c\) is the complement of \(A\).

Let \(s = (p, \{\sigma^*_n\}) \in \tilde{A}^\infty\) and \(B = \{\omega: \tilde{\xi}_{\lambda(p)} \leq \tilde{\xi}_{\lambda(p)}\}\). By Theorem 1 and (1) we have

\[
\widetilde{R}_1(x, s^*, t^*) = R_1(x, p^*, q^*)
\]

\[
\geq E_x \varphi_1(p, q^*)
\]

\[
= E_x g_1(X_{\lambda(p)})1_{\{\lambda(p) < \mu(q^*)\}} + E_x h_1(\mu(q^*), X_{\mu(q^*)})1_{\{\lambda(p) > \mu(q^*)\}}
\]

\[
+ E_x [g_1(X_{\lambda(p)})1_B + h_1(\lambda(p), X_{\lambda(p)})1_{B^c}]1_{\{\lambda(p) = \mu(q^*)\}}
\]
\[= \mathbb{E}_x g_1(X_{\tilde{\lambda}(p)}) \mathbb{I}_{\{\tilde{\lambda}(p) < \mu(q^*)\}} + \mathbb{E}_x \mathbb{E}_{\mu(q^*)} g_1(X_{\sigma^*_{\mu(q^*)}}) \mathbb{I}_{\{\tilde{\lambda}(p) > \mu(q^*)\}} + \mathbb{E}_x \mathbb{E}_{\mu(q^*)} g_1(X_{\sigma^*_{\mu(q^*)}}) \mathbb{I}_{\{\tilde{\lambda}(p) = \mu(q^*)\}}\]

\[= \mathbb{E}_x g_1(X_{\tilde{\lambda}(p)}) \mathbb{I}_{\{\tilde{\lambda}(p) < \mu(q^*)\}} + \mathbb{E}_x \mathbb{E}_{\mu(q^*)} g_1(X_{\sigma^*_{\mu(q^*)}}) \mathbb{I}_{\{\tilde{\lambda}(p) > \mu(q^*)\}} + \mathbb{E}_x \mathbb{E}_{\mu(q^*)} g_1(X_{\sigma^*_{\mu(q^*)}}) \mathbb{I}_{\{\tilde{\lambda}(p) = \mu(q^*)\}}\]

\[\geq \mathbb{E}_x \left[ g_1(X_{\tilde{\lambda}(p)}) \mathbb{I}_{\{\tilde{\lambda}(p) < \mu(q^*)\}} + g_1(X_{\sigma^*_{\mu(q^*)}}) \mathbb{I}_{\{\tilde{\lambda}(p) > \mu(q^*)\}} + \left[ g_1(X_{\tilde{\lambda}(p)}) \mathbb{I}_{\{\tilde{\lambda}(p) = \mu(q^*)\}} \right] \right]\]

\[= \mathbb{E}_x g_1(X_{\tilde{\lambda}}) = \tilde{R}_2(x, \lambda^*, \mu^*) \]

Similar calculations show that for every \( t = (q, \{\sigma^2_n\}) \in \bar{M}_n^N \) we have \( \tilde{R}_2(x, s^*, t^*) \geq \tilde{R}_2(x, s^*, t) \). Hence \((s^*, t^*)\) is an equilibrium point for \( \mathcal{G}_{w_p} \).

In fact, the players play optimally \( \mathcal{G}_{w_p} \) using a Nash equilibrium strategy from \( \mathcal{G}_{w_p} \). If the strategy of both players indicates stopping at moment \( n \) and neither player has stopped earlier, then the lottery chooses one of them. The player who has not been selected will accept any future realization according to the adequate optimal strategy in the optimization problem.

### 3 Two Person Best Choice Problem with Random Priority

The solution of the best choice problem (one player game) is auxiliary in the solution of the two person game with random priority. With sequential observation of the applicants we connect some natural probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The elementary events are the permutation of all applicants and the probability measure \(\mathbb{P}\) is the uniform distribution on \(\Omega\). The observable sequence of relative ranks \(Y_k, k = 1, 2, \ldots, N\) defines the sequence of the \(\sigma\)-fields \(\mathcal{F}_k = \sigma(Y_1, \ldots, Y_k), k = 1, 2, \ldots, N\). The random variables \(Y_k\) are independent and \(P(Y_k = i) = 1/k\). Denote \(\mathcal{F}^N\) the set of all Markov times \(\tau\) with respect to the \(\sigma\)-fields \(\left\{\mathcal{F}_k\right\}_{k=1}^N\) bounded by \(N\). The secretary problem can be formulated as follows: we are searching \(\tau^* \in \mathcal{F}^N\) that

\[\mathbb{P}\{Z_{\tau^*} = 1\} = \sup_{\tau \in \mathcal{F}^N} \mathbb{P}\{Z_{\tau} = 1\}\]
The problem can be reduced to the optimal stopping problem for some homogeneous Markov chain (see Dynkin & Yushkevich (1969)) with suitable payoff functions.

Let \( W_t = 1 \). Define \( W_s = \inf\{r > W_{s-1}; Y_r = 1\}, t > 1, (\inf \emptyset = \infty) \). \((W_t, \mathcal{F}_t, P_{(1,1)})_{t=1}^{N} \) is the homogeneous Markov chain with the state space \( E = \{1, 2, \ldots, N\} \cup \{\infty\}, \mathcal{F}_t = \sigma(W_1, W_2, \ldots, W_t) \) and the following one-step transition probabilities:

\[
\begin{align*}
p(r, s) &= \mathbb{P}\{W_{t+1} = s | W_t = r\} = \frac{r}{s(s-1)} & \text{if } 1 \leq r < s \leq N, \quad p(r, \infty) = 1 - \sum_{s=r+1}^{N} p(r, s), \quad p(\infty, \infty) = 1 \text{ and } 0 \text{ otherwise.}
\end{align*}
\]

The payoff function for the problem defined on \( E \) has a form \( f(r) = \frac{r}{N} \).

Let \( \mathcal{F}^N = \{ \tau \in \mathcal{F}^N: \tau = r \Rightarrow Y_r = 1\} \). It is the set of stopping times with respect to \( \mathcal{F}_t, t = 1, 2, \ldots \). We have \( \mathbb{P}\{Z_{r*} = 1\} = \sup_{\sigma \in \mathcal{F}^N} \mathbb{E}_1 f(W_{a_\sigma}) \) and let \( \sigma^* \) be such that \( \mathbb{P}\{Z_{r*} = 1\} = \mathbb{E}_1 f(W_{a^*_r}) \). Denote \( \check{c}(r) = \sup_{\sigma \geq r} \mathbb{P}\{Z_r = 1\} \).

We have

\[
\check{c}(r) = \begin{cases} c_1(r) & \text{if } r_a \leq r \leq N, \\ c_1(r_a) & \text{if } 1 \leq r < r_a, 
\end{cases}
\]

where \( c_1(r) = \frac{r}{N} \sum_{s=r+1}^{N} \frac{1}{s-1}, \quad r = 1, 2, \ldots, N \) and \( r_a = \inf \left\{ 1 \leq r \leq N: \sum_{s=r+1}^{N} \frac{1}{s-1} \leq 1 \right\} \). When \( N \to \infty \) such that \( \frac{r}{N} \to x \) we obtain \( \check{c}_1(x) = \lim_{N \to \infty} c_1(r) = -x \ln x, \quad a = \lim_{N \to \infty} \frac{r_a}{N} = e^{-1} \approx 0.3679 \) (see Shiryaev (1978), Freeman (1983), Rose (1982)).

Let us consider the two person game with random priority described in Section 2 related to the secretary problem. We admit that both players observe Markov chain \( W_t, t = 1, 2, \ldots \) and their utility functions \( g_j(r) = f(r), j = 1, 2, r \in E \). Let lottery \( \bar{\alpha} \) be constant, i.e. \( \alpha_i = \alpha, \quad i = 1, 2, \ldots, N \). We have \( \tilde{g}_1(r) = \alpha f(r) + (1 - \alpha) \check{c}(r), \tilde{g}_2(r) = (1 - \alpha) f(r) + \alpha \check{c}(r) \) and \( \tilde{g}_r^* = 0 \). Our aim is to determine the equilibria which give the highest and lowest value for Player 1. At first, we construct the highest value Nash equilibrium for Player 1. By analysis of the matrices (3) we have that \( p_r = q_r = 1 \) is an equilibrium point for \( r \geq r_a \). We have then

\[
\check{v}_1(r) = \sum_{i=r+1}^{N} p(r, i) \tilde{g}_1(i)
\]

\[
\check{v}_2(r) = \sum_{i=r+1}^{N} p(r, i) \tilde{g}_2(i)
\]

for \( j = 1, 2 \). For \( r = r_a - 1 \) we have two pure equilibria in (3) in this case: \((1, 0)\) and \((0, 1)\) and one in randomized strategies. Since for \( r < r_a \) we have \( f(r) < \check{c}(r) \)
henceforth we can choose \((1, 0)\) at \(r = r_a - 1\) and assume for induction that the same strategy is optimal for \(r < r_a\). Under this assumption
\[
\tilde{v}_1(r) = \sum_{i=r+1}^{r_a-1} p(r, i) g_1(i) + \sum_{i=r_a}^{N} p(r, i) \tilde{g}_1(i)
\]
\[
\tilde{v}_2(r) = \sum_{i=r+1}^{r_a-1} p(r, i) \tilde{c}(i) + \sum_{i=r_a}^{N} p(r, i) \tilde{g}_2(i).
\]

Since \(f(r)\) is increasing and \(\tilde{c}(r)\) is constant for \(r < r_a\) the strategy \((1, 0)\) can be used as equilibrium in \(r_b \leq r \leq r_a\), where \(r_b = \inf\{r < r_a: \tilde{v}_1(r) \leq g_1(r)\}\). Denote \(r_{b'} = \inf\{r < r_a: \tilde{v}_2(r) \leq g_2(r)\}\). For large \(N\) we have \(r_b < r_{b'}\) if \(\alpha < \alpha_0 = \min\left\{\alpha \in [0, 1]: \frac{2}{2 + \alpha} \geq e^{-(1-\alpha)/2}\right\} \approx 0.5299\). Denote
\[
w_1(r, s, \alpha) = \sum_{i=r+1}^{s-1} p(r, i) f(i) + \sum_{i=s}^{N} p(r, i) \tilde{g}_1(i)
\]
\[
w_2(r, s, \alpha) = \sum_{i=r+1}^{s-1} p(r, i) \tilde{c}(i) + \sum_{i=s}^{N} p(r, i) \tilde{g}_2(i).
\]

For \(\alpha < \alpha_0\) we have
\[
(p^*_r, q^*_r) = \begin{cases} 
(1, 1) & \text{if } r \geq r_a, \\
(1, 0) & \text{if } r_b \leq r < r_a, \\
(0, 0) & \text{if } 1 \leq r < r_b,
\end{cases}
\]

and
\[
v_j(r) = \begin{cases} 
w_j(r, r + 1, \alpha) & \text{if } r \geq r_a, \\
w_j(r, r_a, \alpha) & \text{if } r_b \leq r < r_a, \\
w_j(r_b - 1, r_a, \alpha) & \text{if } 1 \leq r < r_b,
\end{cases}
\]

\(j = 1, 2\). The value of the game is \((v_1, v_2) = (v_1(1), v_2(1))\). When \(N \to \infty\) such that \(\frac{r}{N} \to x\) we obtain
\[ \hat{v}_j(x) = \lim_{N \to \infty} v_j(r) = \begin{cases} 
 \hat{w}_j(x, x, z) & \text{if } x \geq a, \\ 
 \hat{w}_j(x, a, z) & \text{if } b \leq r < a, \\ 
 \hat{w}_j(b, a, z) & \text{if } 0 \leq r < b. 
\end{cases} \]

where

\[ \hat{w}_1(x, y, z) = -x \ln x + (1 - z)x \left( \ln y + \frac{(\ln y)^2}{2} \right), \]

\[ \hat{w}_2(x, y, z) = y - x - (1 - z)x \ln y + az \frac{(\ln y)^2}{2} \]

and \( b = \lim_{N \to \infty} \frac{r_b}{N} = e^{-(3-z)/2}. \) The asymptotic value of the game in this equilibrium is

\[ (\hat{v}_1, \hat{v}_2) = \left( e^{-(3-z)/2}, e^{-1} - \frac{z}{2} e^{-(3-z)/2} \right). \]  

Let \( z \geq a_0. \) Denote

\[ u_1(r, s, t, z) = \sum_{i=r+1}^{s-1} p(r, i) \hat{c}(i) + \sum_{i=r}^{s-1} p(r, i) f(i) + \sum_{i=r_a}^{s} p(r, i) \hat{g}_1(i), \]

\[ u_2(r, s, t, z) = \sum_{i=r+1}^{s-1} p(r, i) f(i) + \sum_{i=r_s}^{s-1} p(r, i) \hat{c}(i) + \sum_{i=r_a}^{s} p(r, i) \hat{g}_2(i). \]

Similar analysis as above leads to conclusion that

\[ (p^*_r, q^*_r) = \begin{cases} 
 (1, 1) & \text{if } r \geq r_a, \\
 (1, 0) & \text{if } b \leq r < r_a, \\
 (0, 1) & \text{if } c \leq r < b, \\
 (0, 0) & \text{if } 1 \leq r < c, 
\end{cases} \]
\[ v_j(r) = \begin{cases} 
    u_j(r, r + 1, r + 1, x) & \text{if } r \geq r_a , \\
    u_j(r, r + 1, r_b, x) & \text{if } r_b \leq r < r_a , \\
    u_j(r_b, r_a, x) & \text{if } r_i \leq r < r_b , \\
    u_j(r_c - 1, r_b, r_a, x) & \text{if } 1 \leq r < r_c .
\end{cases} \]  

(8)

\[ j = 1, 2, \text{ where } r_c = \inf \{ r < r_b; \hat{v}_z(r) \leq g_2(r) \}. \]  

When \( N \to \infty \) such that \( \frac{R}{N} \to x \) we have

\[ \hat{v}_j(x) = \lim_{N \to \infty} v_j(r) = \begin{cases} 
    \hat{u}_j(x, x, x, x) & \text{if } x \geq a , \\
    \hat{u}_j(x, x, a, x) & \text{if } b \leq r < a , \\
    \hat{u}_j(x, b, a, x) & \text{if } c \leq r < b , \\
    \hat{u}_j(c, b, a, x) & \text{if } 0 \leq r < c ,
\end{cases} \]

where \( \hat{u}_1(x, y, z, x) = z - x + \frac{x}{y} \hat{w}_1(y, z, x) \) and \( \hat{u}_2(x, y, z, x) = x \ln \frac{y}{x} + \frac{x}{y} \hat{w}_2(y, z, x) \). The asymptotic value of the game for this equilibrium point is

\[ (\hat{v}_1, \hat{v}_2) = (e^{-1} + e^{-(5/2) + e^{1/2}}(1 - e^{-1/2}), e^{-(5/2) + e^{1/2}}) \]. \]  

(9)

**Theorem 3:** In the random priority two person non-zero sum game of choosing the best applicant the Nash equilibrium which gives the maximal probability of success for Player 1 is given by (4) for \( x < x_0 \) and by (7) for \( x \geq x_0 \). The Nash value for the equilibrium is (5) and (8), respectively. For the limiting case the Nash value is given by (6) and (9), respectively.

Now, we construct the Nash equilibrium with the lowest probability of success for Player 1. The same arguments as above suggest that one can choose \((0, 1)\) in \( r_a - 1 \). Using backward induction procedure as long as possible we minimize the Nash value of Player 1. In such a way we obtain the following equilibrium strategy. For \( x \geq 1 - x_0 \)

\[ (p^*_r, q^*_r) = \begin{cases} 
    (1, 1) & \text{if } r \geq r_a , \\
    (0, 1) & \text{if } r_a \leq r < r_a , \\
    (0, 0) & \text{if } 1 \leq r < r_a .
\end{cases} \]  

(10)
and the Nash value

\[
\nu^*_j(r) = \begin{cases} 
  w^*_j(r, r + 1, \alpha) & \text{if } r \geq r_a , \\
  w^*_j(r, r, \alpha) & \text{if } r_d \leq r < r_a , \\
  w^*_j(r_d - 1, r_a, \alpha) & \text{if } 1 \leq r < r_d , 
\end{cases}
\]

(11)

where \( w^*_j(r, s, \alpha) = w^*_2(r, s, 1 - \alpha) \), \( w^*_2(r, s, \alpha) = w^*_1(r, s, 1 - \alpha) \) and \( r_d = \inf\{r < r_a: v^*_1(r) \leq g_1(r)\} \).

For \( \alpha < 1 - \alpha_0 \) we have

\[
(p^*_r, q^*_s) = \begin{cases} 
  (1, 1) & \text{if } r \geq r_a , \\
  (0, 1) & \text{if } r_d \leq r < r_a , \\
  (1, 0) & \text{if } r_f \leq r < r_d , \\
  (0, 0) & \text{if } 1 \leq r < r_f , 
\end{cases}
\]

(12)

and

\[
\nu^*_j(r) = \begin{cases} 
  u^*_j(r, r + 1, r + 1, \alpha) & \text{if } r \geq r_a , \\
  u^*_j(r, r + 1, r, \alpha) & \text{if } r_d \leq r < r_a , \\
  u^*_j(r, r_d, r, \alpha) & \text{if } r_f \leq r < r_d , \\
  u^*_j(r_f - 1, r_d, r_a, \alpha) & \text{if } 1 \leq r < r_f , 
\end{cases}
\]

(13)

where \( u^*_j(r, s, t, \alpha) = u^*_1(r, s, t, 1 - \alpha) \), \( u^*_1(r, s, t, \alpha) = u^*_1(r, s, t, 1 - \alpha) \) and \( r_f = \inf\{r < r_d: \tilde{v}_1(r) \leq g_1(r)\} \). When \( N \to \infty \) such that \( \frac{r}{N} \to x \) we obtain \( \frac{r_d}{N} \to d = e^{-(2+\alpha)/2} \) and \( \frac{r_f}{N} \to f = e^{-(5/2)+\alpha^2} \). The asymptotic value of the game in this equilibrium is

\[
(\tilde{v}^*_1, \tilde{v}^*_2) = \begin{cases} 
  (e^{-(5/2)+\alpha^2}, e^{-1} + e^{-(5/2)+\alpha^2}(1 - e^{\alpha/2})) & \text{if } \alpha < 1 - \alpha_0 \\
  \left( e^{-1} - \frac{1 - \alpha}{2} e^{-(2+\alpha)/2}, e^{-(2+\alpha)/2} \right) & \text{if } \alpha \geq 1 - \alpha_0 . 
\end{cases}
\]

(14)
Theorem 4: In the random priority two person non-zero sum game of choosing the best applicant the Nash equilibrium which gives the lowest probability of success for Player 1 is given by (10) for $\alpha \geq 1 - \alpha_0$ and by (12) for $\alpha < 1 - \alpha_0$. The Nash value for the equilibrium is (11) and (13), respectively. For limiting case the Nash value is given by (14).

On Fig. 1 one can see the above constructed values of the game as function $\alpha$.

Remark 1: These solutions do not exhaust all Nash points in considered game. The other pure Nash equilibria can be obtained, roughly speaking, by more often “switches” between $(1, 0)$ and $(0, 1)$ strategy (when both strategies are the Nash equilibria in bimatrix game (3)). This idea is used in Remark 3 to construct Nash equilibria, for $\alpha \in [1 - \alpha_0, \alpha_0]$, with equal Nash values for both players (see Fig. 1).

Remark 2: For $\alpha \in (0, 1)$ and for $r$ such that $\tilde{v}_1(r) \leq f(r) < \tilde{c}(r)$ and $\tilde{v}_2(r) \leq f(r) < \tilde{c}(r)$ we can use the randomized Nash equilibrium (see Moulin (1986) for details)

$$
(p^*_r, q^*_r) = \left(\frac{\tilde{v}_2(r) - f(r)}{\tilde{v}_2(r) - f(r) + (1 - \alpha)(f(r) - \tilde{c}(r))}, \frac{\tilde{v}_1(r) - f(r)}{\tilde{v}_1(r) - f(r) + \alpha(f(r) - \tilde{c}(r))}\right).
$$

Fig. 1. The values of the game
Remark 3: Let \( z \in (1 - z_0, z_0) \). We have at least two Nash equilibria with the same Nash values for both players equal \( \exp(-(3 - z_0)/2) \) (in the limiting case). The first pair of strategies is (7) and the second pair is (12) with \( c = f \approx .2908 \) and \( b, d, \) chosen in an appropriate way. The values of \( b \) and \( c \) one can obtain as solution of the system of equation \( \hat{u}(c, b, a, z) = \hat{u}(c, b, a, z) = \exp(-(3 - z_0)/2). \) Similarly, \( d \) and \( f \) is solution of the system of equation \( \hat{u}(f, d, a, z) = \hat{u}(f, d, a, z) = \exp(-(3 - z_0)/2). \) The values of \( b \) and \( d \) for selected \( z \) are given in Table 1.

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<th>( b )</th>
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References


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