UNCERTAIN EMPLOYMENT IN COMPETITIVE BEST CHOICE PROBLEMS

KRZYSZTOF SZAJOWSKI*

Abstract. Mathematical models of competitive selection problems are considered. Two decision makers observe sequentially goods or applicants for a post. These objects are characterized by some numbers. The aim of the players is to accept the most profitable object. The following are rules of acceptance. The object can be accepted only at the moment of its appearance. At each moment \( n \) one object is presented only. When both players want to accept the same object then some rules of assignment applicant to the players are applied. The object accepted by the decision maker is available with some probability only. This is extension of the best choice problem with both uncertain employment and the competitive version of the problem with the priority and the random priority. Several nested Bellman equations are investigated. The solution of examples justifying the model are given.

Key words. optimal stopping problem, game variant, Markov process, random priority, secretary problem, zero sum two person game

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1. Introduction. The paper deal with problems of optimal stopping for a Markov process in a competitive situation. The considered problems are related both to the uncertain employment considered by Smith (1975) and to the competitive optimal stopping problem with priority (see Ferenstein (1992)) or more generally with random priority of the players (see Radzik & Szajowski (1990), Szajowski (1992)). First of all we recall the optimal stopping problem with uncertain employment. Let \( (X_n, \mathcal{F}_n, P_x)_{n=0}^N \) be a homogeneous Markov process defined on probability space \( (\Omega, \mathcal{F}, P) \) with fixed state space \( (E, \mathcal{B}) \). Define the gain function \( f : E \rightarrow \mathbb{R} \). Let \( \mathfrak{M}_N \) be a set of sequences \( \mu = \{\mu_n\}_{n=0}^N \) of \( \{0,1\} \)-valued random variables such that \( \mu_n \) is \( \mathcal{F}_n \)-measurable for every \( n \). Let \( \{\eta_n\}_{n=0}^N \) be a sequence of i.i.d. r.v. with the uniform distribution on \([0,1]\), independent of \( \{X_n\}_{n=0}^N \) and \( \mu \) and let \( \alpha = \{\alpha_n\}_{n=0}^N \) be the sequence of real numbers, \( \alpha_n \in [0,1] \). Define \( \tau_\alpha(\mu) = \inf \{n \geq 0 : \mu_n = 1, \eta_n \leq \alpha \} \).

In the optimal stopping problem with uncertain employment we want to find \( \mu^* \) such that

\[
E_x f(X_{\tau_\alpha(\mu^*)}) = \sup_{\mu \in \mathfrak{M}_N} E_x f(X_{\tau_\alpha(\mu)}) \quad \text{for all} \quad x \in E
\]

and to determine the function \( v(x) = E_x f(X_{\tau_\alpha(\mu^*)}) \). We can look at the above problem as a problem of one decision-maker who wants to accept, on the basis of sequential observation, the most profitable state of the Markov process which appeared in the realization but the solicited state is available with some probability only. The availability is unknown before solicitation. If the decision-maker has made unsuccessful stop he is able to choose any next state under the same rules. The availability is described by the sequence \( \alpha \). The problem was formulated and solved by Smith (1975) as the secretary problem with uncertain employment (see also Yasuda (1983)).

Consider two decision-makers, henceforth called Player 1 and Player 2. Each of them can get at most one of the state from the realization of the Markov chain. Since there is only one random sequence \( \{X_n\}_{n=0}^N \) in a trial, at each instant \( n \) only one player can obtain a realization \( x_n \) of \( X_n \). The realization can be accepted when it appears only. No recall is allowed. We can think of the decision process as an investigation of objects with characteristics described by the Markov process. Both players together can accept at most two objects.

* Institute of Mathematics, Technical University of Wrocław, Wybrzeże Wyspiańskiego 27, PL-50-370 Wrocław, Poland, e-mailszajow@im.pwr.wroc.pl
The problem of assigning the objects to the players when both want to accept the same one can be solved in many ways. Dynkin (1969) assumed that for odd \( n \) Player 1 can choose \( x_n \) and for even \( n \) Player 2 can choose. Other authors solve the problem by more or less arbitrary definition of the payoff function. Sakaguchi (1985) considered some version of the bilateral sequential games related to the no-information secretary problem with uncertain employment. There were investigated the two-person non-zero-sum games with one or two sets of \( N \) objects in the condition of the secretary problem. In the case of one set of objects it can happens that both players attempt to accept the same object. In this case players have half success which is taken into account in the payoff function. Another approach assumes the priority for one decision-maker (see Sakaguchi (1984), Enns & Ferenstein (1985), Radzik & Szajowski (1988), Ravindran & Szajowski (1992)) or the random priority (Fushimi (1981), Radzik & Szajowski (1990), Szajowski (1992)).

We recall briefly the random priority approach to the competitive stopping problem. Let \( f : \mathbb{E}_x \times \mathbb{E}_x \rightarrow \mathbb{R} \) be a \( \mathcal{B} \times \mathcal{B} \) real valued measurable function. Horizon \( N \) is finite. If the players have not accepted previous realizations, they evaluate the state of the Markov chain at instant \( n \) and they have two options, either to accept the observed state of the process at moment \( n \) or to reject it. If both players want to accept the same realization, the following random priority mechanism is applied. Let \( \xi_1, \xi_2, \ldots, \xi_N \) be a sequence of i.i.d. r.v. independent of \( \{X_n\}_{n=0}^N \) with the uniform distribution on \([0,1]\) and \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \) be a given vector of real numbers, \( \alpha_i \in [0,1] \). When both players want to accept realization \( x_n \) of \( X_n \), then Player 1 obtains \( x_n \) if \( \xi_n \leq \alpha_n \), otherwise Player 2 benefits. If one of the players accepts realization \( x_n \) of \( X_n \), then the other one is informed about it and he continues play along. If, in the above decision process, Player 1 and Player 2 have accepted states \( x \) and \( y \), respectively, then Player 2 pays \( f(x,y) \) to Player 1. When only Player 1 (Player 2) accepts state \( x \) \( (y) \) then Player 1 obtains \( f_1(x) = \sup_{y \in \mathcal{E}} f(x,y) \) \( (f_2(y) = \inf_{x \in \mathcal{E}} f(x,y)) \) by assumption. If both players finish the decision process without any accepted state, then they gain 0.

Let \( \mathcal{S}^N \) be the aggregation of Markov times with respect to \( \{F_n\}_{n=0}^N \). We admit that \( \mathbf{P}_x(\tau \leq N) < 1 \) for some \( \tau \in \mathcal{S}^N \) (i.e. there is a positive probability that the Markov chain will not be stopped). The elements of \( \mathcal{S}^N \) are possible strategies for the players with the restriction that Player 2 cannot stop at the same moment as Player 1. If the players declare willingness to accept the same object, the random device decide which player is endowed. Let us formalize these consideration. Denote \( \mathcal{S}_k^N \) \( (\mathcal{S}^N = \mathcal{S}_k^N) \). One can define set of strategies \( \mathcal{A}^N = \{ (\lambda, \{\sigma_n^N\}) : \lambda \in \mathcal{A}^N, \sigma^N_n \in \mathcal{A}^N_{n+1} \ \text{for every} \ n \} \) and \( \mathcal{M}^N = \{ (\mu, \{\sigma_n^M\}) : \mu \in \mathcal{M}^N, \sigma^M_n \in \mathcal{M}^N_{n+1} \ \text{for every} \ n \} \) for Player 1 and 2, respectively. Denote \( \tilde{F}_n = \sigma(F_n, \xi_1, \xi_2, \ldots, \xi_n) \) and let \( \tilde{S}^N \) be the set of stopping times with respect to \( \{\tilde{F}_n\}_{n=0}^N \). Define \( \tau_1 = \lambda \mathbb{1}_{\{\lambda < \mu\}} + (\lambda \mathbb{1}_{\{\lambda \geq \mu\}} + \sigma^N_1 \mathbb{1}_{\{\xi_1 > \alpha_1\}}) \mathbb{1}_{\{\lambda = \mu\}} + \sigma^N_1 \mathbb{1}_{\{\lambda > \mu\}} \) and \( \tau_2 = \mu \mathbb{1}_{\{\lambda > \mu\}} + (\mu \mathbb{1}_{\{\xi_1 = \alpha_1\}} + \sigma^M_1 \mathbb{1}_{\{\xi_1 \leq \alpha_1\}}) \mathbb{1}_{\{\lambda = \mu\}} + \sigma^M_1 \mathbb{1}_{\{\lambda < \mu\}} \). Random variables \( \tau_1 \) and \( \tau_2 \) are Markov times with respect to \( \{\tilde{F}_n\}_{n=0}^N \) and \( \tau_1 \neq \tau_2 \).

Let \( E_x f_1^N(X_m) < \infty \) and \( E_x f_2^N(X_m) < \infty \) for \( n, m = 0,1, \ldots, N \) and \( x \in \mathbb{E} \). Let \( s \in \tilde{A}^N \) and \( t \in \tilde{M}^N \). Define \( \bar{R}(x,s,t) = E_x f(X_{\tau_1}, X_{\tau_2}) \) as the expected gain of Player 1. In this way the normal form of the game \( (\tilde{A}^N, \tilde{M}^N, \bar{R}(x,s,t)) \) is defined. This game is denoted by \( \mathcal{G} \). The game \( \mathcal{G} \) is a model of the considered bilateral stopping problem for the Markov process.

**Definition 1.** Pair \((s^*, t^*)\), \( s^* \in \tilde{A}^N, t^* \in \tilde{M}^N \) is an equilibrium point in the game \( \mathcal{G} \) if for every \( x \in \mathbb{E} \),
\( s \in \tilde{A}^N \) and \( t \in \tilde{M}^N \) we have
\[
\tilde{R}(x, s, t^*) \leq \tilde{R}(x, s^*, t^*) \leq \tilde{R}(x, s^*, t).
\]

The aim is to construct the equilibrium pair \((s^*, t^*)\). Such pair has been constructed in (Szajkowski 1992).

In this paper the competitive approach to the optimal stopping problem admitting the uncertain employment and the random priority is considered. At each moment \( n \) the state of the Markov process \( x_n \) is presented to both players. If the players have not already made an acceptance there are following possibilities.

**Model A**

(i) If only one of them would like to accept the state then he tries to take it. In this moment the random mechanism assign availability to the state (which can depend on the player and the moment of decision \( n \)).

(ii) If both of them are interested in this state then at first the random device chooses the player who will first solicit the state. The availability of the state is similar as in the situation when only one player want to take it.

(iii) If state is not available for player chosen by random device then the observed state at moment \( n \) is lost as in the case when both players reject it. The next state in the sequence is interviewed.

**Model B**

The model differs from Model A only in the case when both players would like to accept the same state. So that points (i) and (ii) are there same.

(iii) If random device chooses Player 1 and the state is not available for (lottery decides about it) him then the observed state at moment \( n \) is solicited by Player 2. The state is available for him as in the situation when only Player 2 tries to take it (the random experiment decides about it). If the state is not available then it is lost and the next state in the sequence is interviewed.

**Model C**

The model differs from Model A and B in the case when both players would like to accept the same state. The points (i) and (ii) of Model A are the same. This model admits that if the state is not available for the player chosen by device then the another player is able to solicited the state.

The Fig. 1 presents Model A. The lottery \( P_L \) assigns the priority to the players. The random devices \( I_L \) and \( II_L \) describe availability of the state to Player 1 and Player 2, respectively. In Model B there is a door between \( I_L \) and \( II_L \) which can be opened from the room \( I_L \). In Model C the door handles are from both sides.

In the next section the normal form of the games described as Model A, B and C will be derived. The solution of the games is given. Section 3 contains examples related to the secretary problem.

2. **Two step random assignment.** Let a homogeneous Markov chain \((X_n, F_n, P_x)_{n=0}^N\) be defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a state space \((\mathbb{E}, \mathcal{B})\) and let \( f : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{R} \) be a \( B \times B \) real valued measurable function. Horizon \( N \) is finite. The players observe the Markov chain and they try to accept the “best realization” according to function \( f \) and a possible selection of another player. Each realization \( x_n \) of \( X_n \) can be accepted by only one player and each player can accept at most one realization. The random priority for the players and
the uncertain employment is assumed. If the players have not accepted previous realizations, they evaluate the state of the Markov chain at instant \( n \) and they have two options, either to solicit the observed state of the process at moment \( n \) or to reject it. If both players want to accept the same realization, the lottery chooses one of them. Let the lottery be described by \( \{ \xi_i \}_{i=0}^{N}, \{ \alpha_i \}_{i=0}^{N} \), where \( \xi_i, i = 0, 1, \ldots \) be a sequence of i.i.d. r.v. with the uniform distribution on \([0, 1] \) and \( \alpha_i, i = 0, 1, \ldots \) are real numbers, \( \alpha_i \in [0, 1] \). When both players want to accept realization \( x_n \) of \( X_n \), then Player 1 obtains \( x_n \) if \( \xi_n \leq \alpha_n \), otherwise Player 2 benefits. The solicited states are available with some probability only. The lottery \( \{ \eta_i \}_{i=0}^{N}, \{ \beta_i \}_{i=0}^{N} \) describes availability of states for \( i \)-th player. When only Player 1 (Player 2) accepts state \( x \) (\( y \)) then Player 1 obtains \( f_1(x) = \sup_{y \in E} f(x, y) \) (\( f_2(y) = \inf_{x \in E} f(x, y) \)) by assumption. If both players finish the decision process without any accepted state, then they gain 0.

Let \( \mathcal{G}^N \) be the aggregation of sequences \( \sigma = \{ \sigma_n \}_{n=0}^{N} \) of \( \{0, 1\} \)-valued random variables such that \( \sigma_n \) is \( \mathcal{F}_n \)-measurable, \( n = 0, 1, \ldots \). If player applies \( \sigma \) then \( \sigma_n = 1 \) means that he declares willingness to accept the realization \( x_n \) of \( X_n \). If \( \sigma_n = 0 \) then the player is not interested in accepting the realization \( x_n \). Denote \( \mathcal{E}_k^N = \{ \sigma \in \mathcal{G}^N : \sigma_0 = 0, \sigma_1 = 0, \ldots, \sigma_{k-1} = 0 \} \). Let \( \mathcal{L}_k^N \) and \( \mathcal{M}_k^N \) be copies of \( \mathcal{E}_k^N \) (\( \mathcal{G}^N = \mathcal{G}_0^N \)). One can define set of strategies \( \tilde{\mathcal{L}}^N = \{ (\lambda, \{ \sigma_n \}) : \lambda \in \mathcal{L}^N, \sigma_n \in \mathcal{L}_{n+1}^N \) for every \( n \) \} and \( \tilde{\mathcal{M}}^N = \{ (\mu, \{ \sigma_n \}) : \mu \in \mathcal{M}^N, \sigma_n \in \mathcal{M}_{n+1}^N \) for every \( n \} \) for Player 1 and 2, respectively. The strategies \( \lambda \) and \( \mu \) are applied to the first acceptance by Player 1 and Player 2, respectively. When the first acceptance occurs at the moment \( n \) and the \( i \)-th player stays alone he uses strategy \( \sigma_n^i, i = 1, 2 \).

Let \( E_x f_1^+(X_n) < \infty \) and \( E_x f_2^- (X_m) < \infty \) for \( n, m = 0, 1, \ldots, N \) and \( x \in E \). Let \( s \in \tilde{\mathcal{L}}^N \) and \( t \in \tilde{\mathcal{M}}^N \). Based on the strategies \( s \) and \( t \) for the players, the definition of lotteries and dependently of the model the expected gains \( \bar{R}_*(x, s, t) \) are obtained. In this way the normal form of the game \( (\tilde{\mathcal{L}}^N, \tilde{\mathcal{M}}^N, \bar{R}_*(x, s, t)) \) is defined. This game is denoted by \( \mathcal{G}^* \). The games \( \mathcal{G}^* \) for the models considered in Section 1 are presented in the following subsections.

**Definition 2.** Pair \((s^*, t^*)\), \( s^* \in \tilde{\mathcal{L}}^N \), \( t^* \in \tilde{\mathcal{M}}^N \) is an equilibrium point in the game \( \mathcal{G}^* \) if for every \( x \in E \), \( s \in \mathcal{L}^N \) and \( t \in \mathcal{M}^N \) we have

\[
\bar{R}_*(x, s, t^*) \leq \bar{R}_*(x, s^*, t^*) \leq \bar{R}_*(x, s^*, t).
\]

The aim is to construct the equilibrium pair \((s^*, t^*)\). The following approach is proposed. When one of
the players accepts realization $x_n$ at moment $n$, the second one will try to maximize his gain without any disturbance from another player like in the optimal stopping with uncertain employment. It means that if there is no acceptance of states till moment $n$, the players must take into account the potential danger from a future decision of the opponent before accepting or rejecting realization $x_n$ of $X_n$. To this end, they consider some auxiliary game $G^*_s$.

Let $s = (\bar{\lambda}, \{\bar{\sigma}^1_n\})$ and $t = (\bar{\mu}, \{\bar{\sigma}^2_n\})$. Define $s_0(x, y) = \beta^1_0 f(x, y) + (1 - \beta^2_0) f_1(x)$, $S_0(x, y) = \beta^1_0 f(x, y) + (1 - \beta^2_0) f_2(y)$ and

$$s_n(x, y) = \inf_{x \in \mathbb{S}^N_{n-1}} E_y f(x, X_{\sigma(1,\beta^2)}),$$
$$S_n(x, y) = \sup_{x \in \mathbb{S}^N_{n-1}} E_x f(X_{\sigma(\bar{\sigma}^1, y)}),$$

for all $x, y \in \mathbb{E}$, $n = 1, 2, \ldots, N$, where $\sigma(\bar{\sigma}^1, y) = \inf\{0 \leq n \leq N : \sigma^1_n = 1, \eta^1_n \leq \beta^1_n\}$ and $\sigma(t, \beta^2) = \inf\{0 \leq n \leq N : \sigma^2_n = 1, \eta^2_n \leq \beta^2_n\}$. By the backward induction Bellman (1957) the function $s_n(x, y)$ ($S_n(x, y)$) can be constructed as $s_n(x, y) = \min\{\beta^2_n f(x, y) + (1 - \beta^2_n) T_2 s_{n-1}(x, y), T_2 s_{n-1}(x, y)\}$ ($S_n(x, y) = \max\{\beta^1_n f(x, y) + (1 - \beta^1_n) T_1 s_{n-1}(x, y), T_1 s_{n-1}(x, y)\}$), where $T_2 f(x, y) = E_y f(x, X_1)$ ($T_1 f(x, y) = E_x f(x, Y_1)$), $T_2$ and $T_1$ preserve measurability. Hence $s_n(x, y)$ ($S_n(x, y)$) are $B \otimes B$ measurable. If Player 1 has accepted $x$ at moment $n$ as the first, then his expected gain is

$$h(n, x) = E_x s_{N-n-1}(x, X_1),$$

for $n = 0, 1, \ldots, N - 1$ and $h(N, x) = f_1(x)$. When Player 2 is the first then the expected gain of Player 1 is

$$H(n, x) = E_x S_{N-n-1}(X_1, x),$$

for $n = 0, 1, \ldots, N - 1$ and $H(N, x) = f_2(x)$. Functions $h(n, x)$ and $H(n, x)$ are well defined. They are $B$-measurable of the second variable, $h(n, X_1)$ and $H(n, X_1)$ are integrable with respect to $P_x$.

Based on the solutions of the optimization problems when player stays along in the decision process we can consider an auxiliary game $G^*_s$. The form of this game depends on the model of assignments when both players want to accept the same state.

2.1. Model A. Let us assume that the random priority and the uncertain employment are used as Model A described in Section 1. The sets of strategies for Player 1 and Player 2 are $\mathcal{L}^N$ and $\mathcal{M}^N$, respectively. For $s = (\lambda, \{\sigma^1_n\}) \in \mathcal{L}^N$ and $t = (\mu, \{\sigma^2_n\}) \in \mathcal{M}^N$ we define random variables

$$\lambda_{\alpha, \beta^1}(s, t) = \inf\{0 \leq n \leq N : \lambda_n = 1, \mu_n = 1, \xi_n \leq \alpha_n, \eta^1_n \leq \beta^1_n\},$$

or $\lambda_n = 1, \mu_n = 0, \eta^1_n \leq \beta^1_n$, (3)

$$\mu_{\alpha, \beta^2}(s, t) = \inf\{0 \leq n \leq N : \lambda_n = 1, \mu_n = 1, \xi_n > \alpha_n, \eta^2_n \leq \beta^2_n\},$$

or $\lambda_n = 0, \mu_n = 1, \eta^2_n \leq \beta^2_n$. (4)

Let

$$\tau_1(s, t) = \lambda_{\alpha, \beta^1}(s, t) I_{\{\alpha, \beta^1(\tau_1) < \sigma_{\alpha, \beta^1}(s, t)\}} + \sigma_{\alpha, \beta^2}(s, t) I_{\{\lambda_{\alpha, \beta^1}(s, t) > \sigma_{\alpha, \beta^2}(s, t)\}}$$
and
\[
\tau_2(s, t) = \mu_{\alpha, \beta^2}(s, t)\mathbb{I}_{\{\lambda_{\alpha, \beta^1}(s, t) > \mu_{\alpha, \beta^2}(s, t)\}} + \sigma_{\lambda_{\alpha, \beta^1}(s, t) < \mu_{\alpha, \beta^2}(s, t)},
\]
where \(\mathbb{I}_A\) is a characteristic function of set \(A\). We have \(\tilde{\Theta}_A(x, s, t) = E_x f(X_{\tau_1}(s, t), X_{\tau_2}(s, t))\).

In the auxiliary game \(G_X^A\) the sets of strategies for Player 1 and Player 2 are \(\mathcal{L}^N\) and \(\mathcal{M}^N\), respectively. For \(\lambda \in \mathcal{L}^N\) and \(\bar{\mu} \in \mathcal{M}^N\) we define the random variables
\[
\lambda_{\alpha, \beta^1}(\lambda, \bar{\mu}) = \inf\{0 \leq n \leq N : \lambda_n = 1, \mu_n = 1, \xi_n = \alpha_n, \eta_n \leq \beta^1_n
\]
or \(\lambda_n = 1, \mu_n = 0, \eta_n \leq \beta^1_n\},
\]
\[
\mu_{\alpha, \beta^2}(\lambda, \bar{\mu}) = \inf\{0 \leq n \leq N : \lambda_n = 1, \mu_n = 1, \xi_n > \alpha_n, \eta_n \leq \beta^2_n
\]
or \(\lambda_n = 0, \mu_n = 1, \eta_n \leq \beta^2_n\}.
\]
Define payoff function
\[
r(\lambda_{\alpha, \beta^1}(\lambda, \bar{\mu}), \mu_{\alpha, \beta^2}(\lambda, \bar{\mu}))
\]
\[\begin{align*}
&= \left\{ \begin{array}{ll}
& h(\lambda_{\alpha, \beta^1}(\lambda, \bar{\mu}), X_{\lambda_{\alpha, \beta^1}(\lambda, \bar{\mu})}, \mu_{\alpha, \beta^2}(\lambda, \bar{\mu}))
& \mathbb{I}_{\{\lambda_{\alpha, \beta^1}(\lambda, \bar{\mu}) < \mu_{\alpha, \beta^2}(\lambda, \bar{\mu})\}}
& \text{if } \lambda_{\alpha, \beta^1}(\lambda, \bar{\mu}) \leq N
\end{array} \right.

&+ H(\mu_{\alpha, \beta^2}(\lambda, \bar{\mu}), X_{\mu_{\alpha, \beta^2}(\lambda, \bar{\mu})}, \lambda_{\alpha, \beta^1}(\lambda, \bar{\mu}))
& \mathbb{I}_{\{\lambda_{\alpha, \beta^1}(\lambda, \bar{\mu}) > \mu_{\alpha, \beta^2}(\lambda, \bar{\mu})\}}
& \text{if } \mu_{\alpha, \beta^2}(\lambda, \bar{\mu}) \leq N

&0
& \text{otherwise},
\end{align*}\]
As a solution of the game we search for equilibrium pair \((\lambda^*, \mu^*)\) such that
\[
\mathcal{R}(x, \lambda_{\alpha, \beta^1}(\lambda^*, \mu^*), \mu_{\alpha, \beta^2}(\lambda^*, \mu^*)) \leq \mathcal{R}(x, \lambda_{\alpha, \beta^1}, \mu_{\alpha, \beta^2}(\lambda, \mu))
\]
for all \(x \in \mathbb{E}\), where \(\mathcal{R}(x, \lambda_{\alpha, \beta^1}(\lambda, \mu), \mu_{\alpha, \beta^2}(\lambda, \mu)) = E_x r(\lambda_{\alpha, \beta^1}(\lambda, \mu), \mu_{\alpha, \beta^2}(\lambda, \mu))\).

Because the Markov process is observed here, one can define a sequence \(v_n(x)\), \(n = 0, 1, \ldots, N + 1\) on \(E\) by setting \(v_{N+1}(x) = 0\) and
\[
v_n(x) = \text{val}
\[
\begin{bmatrix}
\alpha_n(\beta^1_n h(n, x) + (1 - \beta^1_n)Tv_{n+1}(x)) & \beta^1_n h(n, x) + (1 - \beta^1_n)Tv_{n+1}(x) \\
(+1 - \alpha_n)(\beta^2_n H(n, x) + (1 - \beta^2_n)Tv_{n+1}(x)) & \beta^2_n H(n, x) + (1 - \beta^2_n)Tv_{n+1}(x)
\end{bmatrix}
\]
for \(n = 0, 1, \ldots, N\), where \(Tv_*(x) = E_x v_*(X_1)\) and \(\text{val} A\) denotes a value of the two person zero-sum game with payoff matrix \(A\) (see Luce & Raiffa (1957), Yasuda (1985)).

To prove the correctness of the construction let us observe that the payoff matrix in (9) has the form
\[
A = \begin{bmatrix}
s & f \\
(\alpha - b)a + b & a \\
b & c
\end{bmatrix},
\]
where \(a, b, c, \alpha\) are real numbers and \(\alpha \in [0, 1]\). By direct checking we have

**Lemma 2.1.** *The two person zero-sum game with payoff matrix \(A\) given by (10) has an equilibrium point \((\epsilon, \delta)\) in pure strategies, where*
\[
(\epsilon, \delta) = \begin{cases}
(s, s) & \text{if } a \geq b, \\
(s, f) & \text{if } c \leq a < b, \\
(f, s) & \text{if } a < b \leq c, \\
(f, f) & \text{if } a < c < b.
\end{cases}
\]
Notice that $v_{N+1}$ is measurable. Let us assume that functions $v_i$, $i = N, N-1, \ldots, n+1$ are measurable. Denote

$$A_n^a = \{x \in E : \beta_n^1 h(n, x) - \beta_n^2 H(n, x) \geq (\beta_n^1 - \beta_n^2)Tv_{n+1}(x)\}$$

$$A_n^{af} = \{x \in E : \beta_n^1 h(n, x) - \beta_n^2 H(n, x) < (\beta_n^1 - \beta_n^2)Tv_{n+1}(x), h(n, x) \geq Tv_{n+1}(x)\}$$

$$A_n^b = \{x \in E : \beta_n^1 h(n, x) - \beta_n^2 H(n, x) < (\beta_n^1 - \beta_n^2)Tv_{n+1}(x), h(n, x) \leq Tv_{n+1}(x)\}$$

and

$$A_n^f = \mathbb{E} \setminus (A_n^a \cup A_n^{af} \cup A_n^b).$$

By the definition of the sets $A_n^a, A_n^{af}, A_n^b \in \mathcal{B}$ and Lemma 2.1 we have

$$v_n(x) = [\alpha_n \beta_n^1 (h(n, x) - Tv_{n+1}(x)) + (1 - \alpha_n) \beta_n^2 (H(n, x) - Tv_{n+1}(x))]A_n^a(x)$$

$$+ \alpha_n \beta_n^1 (h(n, x) - Tv_{n+1}(x))A_n^{af}(x)$$

$$+ \beta_n^2 (H(n, x) - Tv_{n+1}(x))A_n^b(x) + Tv_{n+1}(x),$$

hence $v_n(x)$ is $\mathcal{B}$-measurable.

Define

$$\lambda_n^* = \begin{cases} 1 & \text{if } X_n \in A_n^a \cup A_n^{af} \\ 0 & \text{otherwise.} \end{cases}$$

$$\mu_n^* = \begin{cases} 1 & \text{if } X_n \in A_n^a \cup A_n^b \\ 0 & \text{otherwise.} \end{cases}$$

**Theorem 2.2.** Game $\mathcal{G}_a^A$ with payoff function (7) and the sets of strategies $\mathcal{S}_N^N$ and $\mathcal{M}_N$ for Player 1 and 2, respectively, has a solution. Pair $(\lambda^*, \mu^*)$ defined by (11) and (12) is the equilibrium point and $v_0(x)$ is the value of the game.

Let us construct an equilibrium pair $(s^*, t^*)$ for game $\mathcal{G}_a^A$. Define (see Smith (1975) and Yasuda (1983))

$$\sigma_{n,m}^{1*} = \begin{cases} 1 & \text{if } s_{N-m}(X_m, X_n) = f(X_m, X_n), \\ 0 & \text{if } s_{N-m}(X_m, X_n) > f(X_m, X_n), \end{cases}$$

$$\sigma_{n,m}^{2*} = \begin{cases} 1 & \text{if } s_{N-m}(X_m, X_n) = f(X_m, X_n), \\ 0 & \text{if } s_{N-m}(X_m, X_n) < f(X_m, X_n), \end{cases}$$

Let $(\lambda^*, \mu^*)$ be an equilibrium point in $\mathcal{G}_a^A$.

**Theorem 2.3.** Game $\mathcal{G}_a^A$ has a solution. Pair $(s^*, t^*)$ such that $s^* = (\lambda^*, \{\sigma_{n}^{1*}\})$ and $t^* = (\mu^*, \{\sigma_{n}^{2*}\})$ is the equilibrium point. The value of the game is $v_0(x)$.

### 2.2. Model B

Let us assume that the random priority and the uncertain employment are used as in described in Section 1 Model B. The game related to Model B we denote $\mathcal{G}_b^B$. The sets of strategies for Player 1 and Player 2 are $\mathcal{S}_N^N$ and $\mathcal{M}_N$, respectively. For $s = (\lambda, \{\sigma_{n}^{1}\}) \in \mathcal{S}_N^N$ and $t = (\mu, \{\sigma_{n}^{2}\}) \in \mathcal{M}_N$ we define random variables

$$\lambda_{\alpha, \beta_1}(s, t) = \inf\{0 \leq n \leq N : \lambda_n = 1, \mu_n = 1, \xi_n \leq \alpha_n, \eta_n \leq \beta_n^1 \}$$

or $\lambda_n = 1, \mu_n = 0, \eta_n \leq \beta_n^1$, (15)

$$\mu_{\alpha, \beta_1}(s, t) = \inf\{0 \leq n \leq N : \lambda_n = 1, \mu_n = 1, \xi_n > \alpha_n, \eta_n \leq \beta_n^2 \}$$

or $\lambda_n = 0, \mu_n = 1, \xi_n \leq \alpha_n, \eta_n \leq \beta_n^2$ (16)
Let
\[ \tau_1(s,t) = \lambda_{\alpha,\beta_1}(s,t)\mathbb{I}_{\{\lambda_{\alpha,\beta_1}(s,t) < \mu_{\alpha,\beta_1}(s,t)\}} + \sigma_{\alpha,\beta_1}(s,t)\mathbb{I}_{\{\lambda_{\alpha,\beta_1}(s,t) > \mu_{\alpha,\beta_1}(s,t)\}} \]
and
\[ \tau_2(s,t) = \mu_{\alpha,\beta_2}(s,t)\mathbb{I}_{\{\lambda_{\alpha,\beta_2}(s,t) < \mu_{\alpha,\beta_2}(s,t)\}} + \sigma_{\alpha,\beta_2}(s,t)\mathbb{I}_{\{\lambda_{\alpha,\beta_2}(s,t) > \mu_{\alpha,\beta_2}(s,t)\}}. \]

We have \( \mathcal{R}_B(x,s,t) = E_x f(X_{\tau_1}(s,t), X_{\tau_2}(s,t)) \).

In the auxiliary game \( G^B_2 \), the sets of strategies for Player 1 and Player 2 are \( \mathcal{L}^N \) and \( \mathcal{M}^N \), respectively. For \( \lambda \in \mathcal{L}^N \) and \( \mu \in \mathcal{M}^N \) we define the random variables
\[
\tilde{\lambda}_{\alpha,\beta_1}(\lambda, \mu) = \inf \{ 0 \leq n \leq N : \lambda_n = 1, \mu_n = 1, \xi_n \leq \alpha_n, \eta_n \leq \beta_1^n \}
\]
or
\[
\lambda_n = 0, \mu_n = 1, \xi_n > \alpha_n, \eta_n \leq \beta_1^n \}
\]
\[ \tilde{\mu}_{\alpha,\beta_2}(\lambda, \mu) = \inf \{ 0 \leq n \leq N : \lambda_n = 1, \mu_n = 1, \xi_n \leq \alpha_n, \eta_n \leq \beta_2^n \}
\]
or
\[
\lambda_n = 0, \mu_n = 1, \eta_n > \beta_2^n \}
\]

Define payoff function
\[
r(\tilde{\lambda}_{\alpha,\beta_1}(\lambda, \mu), \tilde{\mu}_{\alpha,\beta_2}(\lambda, \mu)) = \begin{cases} 
  h(\tilde{\lambda}_{\alpha,\beta_1}(\lambda, \mu), \tilde{\mu}_{\alpha,\beta_2}(\lambda, \mu)) & \text{if } \tilde{\lambda}_{\alpha,\beta_1}(\lambda, \mu) \leq N \\
  +H(\tilde{\mu}_{\alpha,\beta_2}(\lambda, \mu), \tilde{\mu}_{\alpha,\beta_2}(\lambda, \mu)) & \text{if } \tilde{\mu}_{\alpha,\beta_2}(\lambda, \mu) \leq N \\
  0 & \text{otherwise},
\end{cases}
\]

As a solution of the game we search for equilibrium pair \((\tilde{\lambda}^*, \tilde{\mu}^*)\) such that
\[
\mathcal{R}(x, \tilde{\lambda}_{\alpha,\beta_1}(\lambda, \mu), \tilde{\mu}_{\alpha,\beta_2}(\lambda, \mu)) \leq \mathcal{R}(x, \tilde{\lambda}_{\alpha,\beta_1}(\lambda^*, \mu^*), \tilde{\mu}_{\alpha,\beta_2}(\lambda^*, \mu^*))
\]
for all \( x \in \mathbb{E} \), where \( \mathcal{R}(x, \tilde{\lambda}_{\alpha,\beta_1}(\lambda, \mu), \tilde{\mu}_{\alpha,\beta_2}(\lambda, \mu)) = E_x r(\tilde{\lambda}_{\alpha,\beta_1}(\lambda, \mu), \tilde{\mu}_{\alpha,\beta_2}(\lambda, \mu)) \).

We can define, similarly as for Model A, a sequence \( v_n(x) \), \( n = 0, 1, \ldots, N + 1 \) on \( \mathbb{E} \) by setting \( v_{N+1}(x) = 0 \) and
\[
v_n(x) = \text{val} \begin{bmatrix}
\alpha_n \beta_1^1 h(n,x) + (1 - \alpha_n \beta_1^n)(\beta_2^1 H(n,x) - \beta_2^n h(n,x) + (1 - \beta_2^n)Tv_{n+1}(x)) \\
+ (1 - \beta_2^n)Tv_{n+1}(x) \\
\beta_2^1 H(n,x) - \beta_2^n h(n,x) + (1 - \beta_2^n)Tv_{n+1}(x) \\
Tv_{n+1}(x)
\end{bmatrix}
\]
for \( n = 0, 1, \ldots, N \).

To prove the correctness of the construction let us observe that the payoff matrix in (21) after equivalent transformation has the form
\[
A = \begin{bmatrix}
s & f \\
\alpha a + (1 - \alpha \beta) b & a \\
b & 0
\end{bmatrix},
\]
where \( a, b, \alpha, \beta \) are real numbers and \( \alpha, \beta \in [0, 1] \). By direct checking we have
LEMMA 2.4. The two person zero-sum game with payoff matrix $A$ given by (22) has an equilibrium point $(\epsilon, \delta)$ in pure strategies, where

$$(\epsilon, \delta) = \begin{cases} (s,s) & \text{if } a \geq \beta b \text{ and } (1 - \alpha \beta) b \leq (1 - \alpha) a, \\ (s,f) & \text{if } a \geq 0 \text{ and } (1 - \alpha \beta) b > (1 - \alpha) a, \\ (f,s) & \text{if } b \leq 0 \text{ and } a < \beta b, \\ (f,f) & \text{if } a > 0 \text{ and } b < 0. \end{cases}$$

Notice that $v_{N+1}$ is measurable. Let us assume that functions $v_i$, $i = N, N-1, \ldots, n+1$ are measurable. Denote

$$(23) \quad A_n^{ss} = \{ x \in \mathbb{E} : (1 - \alpha_n) \beta_n^1(h(n, x) - Tr_{n+1}(x)) \geq (1 - \alpha_n \beta_n^1 \beta_n^2(H(n, x) - Tr_{n+1}(x)), \beta_n^2(H(n, x) - Tr_{n+1}(x)) \leq h(n, x) - Tr_{n+1}(x) \}$$

$$(24) \quad A_n^{sf} = \{ x \in \mathbb{E} : h(n, x) \geq Tr_{n+1}(x), (1 - \alpha_n) \beta_n^1(h(n, x) - Tr_{n+1}(x)) < (1 - \alpha_n \beta_n^1 \beta_n^2(H(n, x) - Tr_{n+1}(x)) \}$$

$$(25) \quad A_n^{fs} = \{ x \in \mathbb{E} : H(n, x) \leq Tr_{n+1}(x), h(n, x) - \beta_n^2 H(n, x) \leq (1 - \beta_n^2) Tr_{n+1}(x) \}$$

and

$$(26) \quad A_n^{ff} = \mathbb{E} \setminus (A_n^{ss} \cup A_n^{sf} \cup A_n^{fs}).$$

By the definition of the sets $A_n^{ss}, A_n^{sf}, A_n^{fs} \in \mathbb{R}$ and Lemma 2.4 we have

$$(27) \quad v_n(x) = \alpha_n \beta_n^1(h(n, x) - Tr_{n+1}(x)) + (1 - \alpha_n \beta_n^1 \beta_n^2(H(n, x) - Tr_{n+1}(x)) || A_n^{ss}(x) + \alpha_n \beta_n^2(h(n, x) - Tr_{n+1}(x)) || A_n^{sf}(x) + \beta_n^2(H(n, x) - Tr_{n+1}(x)) || A_n^{fs}(x) + Tr_{n+1}(x).$$

The stopping times $\lambda_n^*$ and $\mu_n^*$ are defined by (11) and (12), respectively, with appropriate $A_n^*$.  

THEOREM 2.5. Game $G_n^B$ with payoff function (19) and sets of strategies $\mathcal{L}^N$ and $\mathcal{M}^N$ for Player 1 and 2, respectively, has a solution. Pair $(\lambda^*, \mu^*)$ defined by (11) and (12), based on (23-26), is the equilibrium point and $v_0(x)$ is the value of the game.

Let us construct an equilibrium pair $(s^*, t^*)$ for game $G_n^B$. Let $(\lambda^*, \mu^*)$ be an equilibrium point in $G_n^B$. 

THEOREM 2.6. Game $G_n^B$ has a solution. Pair $(s^*, t^*)$ such that $s^* = (\lambda^*, \{\sigma_n^1\})$ and $t^* = (\mu^*, \{\sigma_n^2\})$ is the equilibrium point, where $(\lambda^*, \mu^*)$ be an equilibrium point in $G_n^B$ and the strategies $\{\sigma_n^1\}$ and $\{\sigma_n^2\}$ are defined by (13) and (14), respectively. The value of the game is $v_0(x)$, where $v_n(x)$ is given by (27).

2.3 Model C. Let us assume that the random priority and the uncertain employment are used as in described in Section 1 Model C. The game related to Model C we denote $G_n^C$. The sets of strategies for Player 1 and Player 2 are $\mathcal{L}^N$ and $\mathcal{M}^N$, respectively. For $s = (\lambda, \{\sigma_n^1\}) \in \mathcal{L}^N$ and $t = (\mu, \{\sigma_n^2\}) \in \mathcal{M}^N$ we define random variables

$$(28) \quad \lambda_{\alpha, \beta_1, \beta_2}(s, t) = \inf \{ 0 \leq n \leq N : \lambda_n = 1, \mu_n = 1, \xi_n \leq \alpha_n, \eta_n^1 \leq \beta_n^1 \}
\quad \text{or } \lambda_n = 1, \mu_n = 0, \eta_n^1 \leq \beta_n^1
\quad \text{or } \lambda_n = 0, \mu_n = 1, \eta_n^1 \leq \beta_n^1, \eta_n^2 > \beta_n^2 \},$$

$$(29) \quad \mu_{\alpha, \beta_1, \beta_2}(s, t) = \inf \{ 0 \leq n \leq N : \lambda_n = 1, \mu_n = 1, \xi_n > \alpha_n, \eta_n^2 \leq \beta_n^2 \}
\quad \text{or } \lambda_n = 0, \lambda_n = 1, \mu_n = 1, \xi_n \leq \alpha_n, \eta_n^1 < \beta_n^1, \eta_n^2 \leq \beta_n^2
\quad \text{or } \lambda_n = 0, \mu_n = 1, \eta_n^2 \leq \beta_n^2 \}. $$
Let
\[
\tau_1(s, t) = \lambda_{\alpha, \beta_1, \beta_2}(s, t)I_{\{\lambda_{\alpha, \beta_1}, \beta_2(s, t) < \mu_{\alpha, \beta_1, \beta_2}(s, t)\}} + \sigma_{\mu_{\alpha, \beta_1, \beta_2}(s, t), \beta_1}I_{\{\lambda_{\alpha, \beta_1, \beta_2}(s, t) > \mu_{\alpha, \beta_1, \beta_2}(s, t)\}}
\]
and
\[
\tau_2(s, t) = \mu_{\alpha, \beta_1, \beta_2}(s, t)I_{\{\lambda_{\alpha, \beta_1}, \beta_2(s, t) > \mu_{\alpha, \beta_1, \beta_2}(s, t)\}} + \sigma_{\lambda_{\alpha, \beta_1, \beta_2}(s, t), \beta_2}I_{\{\lambda_{\alpha, \beta_1, \beta_2}(s, t) < \mu_{\alpha, \beta_1, \beta_2}(s, t)\}}.
\]

We have \(\tilde{R}_N(x, s, t) = E_x f(X_{r_1}(s, t), X_{r_2}(s, t))\).

In the auxiliary game \(G^N_a\) the sets of strategies for Player 1 and Player 2 are \(\mathcal{L}^N\) and \(\mathcal{M}^N\), respectively. For \(\lambda \in \mathcal{L}^N\) and \(\mu \in \mathcal{M}^N\) we define the random variables
\begin{align}
\tilde{\lambda}_{\alpha, \beta_1, \beta_2}(\lambda, \mu) &= \inf \{0 \leq n \leq N : \lambda_n = 1, \mu_n = 1, \xi_n \leq \alpha_n, \eta_n^1 \leq \beta_1^1 \text{ or } \lambda_n = 1, \mu_n = 0, \eta_n^1 \leq \beta_1^2 \} \\
\text{or } \lambda_n = 0, \mu_n = 1, \eta_n^1 \leq \beta_1^1, \eta_n^1 > \beta_1^2, \\
\tilde{\mu}_{\alpha, \beta_1, \beta_2}(\lambda, \mu) &= \inf \{0 \leq n \leq N : \lambda_n = 1, \mu_n = 1, \xi_n \leq \alpha_n, \eta_n^2 \leq \beta_2^1 \text{ or } \lambda_n = 0, \mu_n = 1, \eta_n^2 \leq \beta_2^2 \}.
\end{align}

Define payoff function
\begin{align}
\tilde{r}(\lambda_{\alpha, \beta_1, \beta_2}(\lambda, \mu), \tilde{\mu}_{\alpha, \beta_1, \beta_2}(\lambda, \mu)) &= \begin{cases} 
H(\tilde{\mu}_{\alpha, \beta_1, \beta_2}(\lambda, \mu), \tilde{\lambda}_{\alpha, \beta_1, \beta_2}(\lambda, \mu))I_{\{\lambda_{\alpha, \beta_1}, \beta_2(\lambda, \mu) < \mu_{\alpha, \beta_1, \beta_2}(\lambda, \mu)\}} & \text{if } \tilde{\lambda}_{\alpha, \beta_1, \beta_2}(\lambda, \mu) \leq N \\
0 & \text{or } \tilde{\lambda}_{\alpha, \beta_1, \beta_2}(\lambda, \mu) \leq N \\
+H(\tilde{\mu}_{\alpha, \beta_1, \beta_2}(\lambda, \mu), \tilde{\lambda}_{\alpha, \beta_1, \beta_2}(\lambda, \mu))I_{\{\lambda_{\alpha, \beta_1}, \beta_2(\lambda, \mu) > \mu_{\alpha, \beta_1, \beta_2}(\lambda, \mu)\}} & \text{otherwise},
\end{cases}
\end{align}

As a solution of the game we search for equilibrium pair \((\lambda^*, \mu^*)\) such that
\begin{align}
\mathcal{R}(x, \tilde{\lambda}_{\alpha, \beta_1, \beta_2}(\lambda, \mu), \tilde{\mu}_{\alpha, \beta_1, \beta_2}(\lambda, \mu)) &\leq \mathcal{R}(x, \lambda^*, \lambda^*, \tilde{\mu}_{\alpha, \beta_1, \beta_2}(\lambda, \mu)) \\
\leq \mathcal{R}(x, \lambda^*, \lambda^*, \tilde{\mu}_{\alpha, \beta_1, \beta_2}(\lambda, \mu)) &\leq \mathcal{R}(x, \lambda^*, \lambda^*, \tilde{\mu}_{\alpha, \beta_1, \beta_2}(\lambda, \mu))
\end{align}
for all \(x \in \mathbb{E}\), where \(\mathcal{R}(x, \tilde{\lambda}_{\alpha, \beta_1, \beta_2}(\lambda, \mu), \tilde{\mu}_{\alpha, \beta_1, \beta_2}(\lambda, \mu)) = E_x \tilde{r}(\tilde{\lambda}_{\alpha, \beta_1, \beta_2}(\lambda, \mu), \tilde{\mu}_{\alpha, \beta_1, \beta_2}(\lambda, \mu))\).

We can define, similarly as for Model A and Model B, a sequence \(v_n(x), n = 0, 1, \ldots, N + 1\) on \(\mathbb{E}\) by setting \(v_{N+1}(x) = 0\) and
\begin{align}
v_n(x) = \begin{bmatrix}
\alpha_n(\beta_1^1 h(n, x) + (1 - \beta_1^1)g(n, x, \beta_2^1)) + (1 - \alpha_n)(\beta_2^1 H(n, x) + (1 - \beta_2^1)G(n, x, \beta_1^1)) \\
\alpha_n(\beta_1^1 h(n, x) + (1 - \beta_1^1)g(n, x, \beta_2^1)) + (1 - \alpha_n)(\beta_2^1 H(n, x) + (1 - \beta_2^1)G(n, x, \beta_1^1)) \end{bmatrix}
\end{align}
for \(n = 0, 1, \ldots, N\), where \(G(n, x, \beta_1^1) = \beta_1^2 h(n, x) + (1 - \beta_1^1)T v_{n+1}(x)\) and \(g(n, x, \beta_2^1) = \beta_2^1 H(n, x) + (1 - \beta_2^1)T v_{n+1}(x)\).

To prove the correctness of the construction let us observe that the payoff matrix in (34), after equivalent transformation, has the form
\begin{align}
A = \begin{bmatrix}
s & f \\
\alpha(a + (1 - \beta)b) & a \\
+ (1 - \alpha)(b + (1 - \gamma)a) & b \\
\end{bmatrix}
\end{align}
where \(a, b, \alpha, \beta, \gamma\) are real numbers and \(\alpha, \beta, \gamma \in [0, 1]\). By direct checking we have
Lemma 2.7. The two person zero-sum game with payoff matrix $A$ given by (22) has an equilibrium point $(\epsilon, \delta)$ in pure strategies, where

$$
(\epsilon, \delta) = \begin{cases} 
(s, s) & \text{if } (1 - (1 - \alpha)\gamma)a \geq \alpha \beta b \text{ and } (1 - \alpha)\beta b \leq (1 - \alpha)\gamma a, \\
(s, f) & \text{if } a \geq 0 \text{ and } (1 - \alpha)\beta b > (1 - \alpha)\gamma a, \\
(f, s) & \text{if } b \leq 0 \text{ and } (1 - (1 - \alpha)\gamma)a \leq \alpha \beta b, \\
(f, f) & \text{if } a > 0 \text{ and } b < 0.
\end{cases}
$$

Denote

$$
A_{n}^{ss} = \{ x \in E : (1 - (1 - \alpha_n)\beta_n^2)(h(n, x) - Tv_{n+1}(x)) \geq \alpha_n\beta_n^2(H(n, x) - Tv_{n+1}(x)), \\
(1 - \alpha_n\beta_n^1)(H(n, x) - Tv_{n+1}(x)) \leq (1 - \alpha_n)\beta_n^1(h(n, x) - Tv_{n+1}(x)) \}
$$

$$
A_{n}^{sf} = \{ x \in E : h(n, x) \geq Tv_{n+1}(x), (1 - \alpha_n)\beta_n^1(h(n, x) - Tv_{n+1}(x)) \\
< (1 - \alpha_n\beta_n^1)\beta_n^2(H(n, x) - Tv_{n+1}(x)) \}
$$

$$
A_{n}^{fs} = \{ x \in E : H(n, x) \leq Tv_{n+1}(x), \\
(1 - (1 - \alpha_n))(h(n, x) - Tv_{n+1}(x)) < \alpha_n(H(n, x) - Tv_{n+1}(x)) \}
$$

and

$$
A_{n}^{ff} = E \setminus (A_{n}^{ss} \cup A_{n}^{sf} \cup A_{n}^{fs}).
$$

By the definition of the sets $A_{n}^{ss}, A_{n}^{sf}, A_{n}^{fs} \in B$ and Lemma 2.7 we have

$$
v_n(x) = \left[ \alpha_n(\beta_n^1(h(n, x) - Tv_{n+1}(x)) + (1 - \beta_n^1)\beta_n^2(H(n, x) - Tv_{n+1}(x)) \\
+ (1 - \alpha_n)(\beta_n^2(H(n, x) - Tv_{n+1}(x)) + (1 - \beta_n^2)\beta_n^1(h(n, x) - Tv_{n+1}(x))) \right] A_{n}^{ss}(x) \\
+ \beta_n^1(h(n, x) - Tv_{n+1}(x)) A_{n}^{sf}(x) \\
+ \beta_n^2(H(n, x) - Tv_{n+1}(x)) A_{n}^{fs}(x) + Tv_{n+1}(x).
$$

The stopping times $\lambda_n^*$ and $\mu_n^*$ are defined by (11) and (12), respectively, with appropriate $A_n^*$ given by (36-39).

Theorem 2.8. Game $G_n^C$ with payoff function (32) and sets of strategies $\mathcal{L}_n^N$ and $\mathcal{R}_n^N$ for Player 1 and 2, respectively, has a solution. Pair $(\lambda^*, \mu^*)$ defined by (11) and (12), based on (23-26), is the equilibrium point and $v_0(x)$ is the value of the game.

Let us construct an equilibrium pair $(s^*, t^*)$ for game $G_n^C$. Let $(\lambda^*, \mu^*)$ be an equilibrium point in $G_n^C$.

Theorem 2.9. Game $G_n^C$ has a solution. Pair $(s^*, t^*)$ such that $s^* = (\lambda^*, \{ \sigma_n^1 \})$ and $t^* = (\mu^*, \{ \sigma_n^2 \})$ is the equilibrium point, where $(\lambda^*, \mu^*)$ be an equilibrium point in $G_n^C$ and the strategies $\{ \sigma_n^1 \}$ and $\{ \sigma_n^2 \}$ are defined by (13) and (14), respectively. The value of the game is $v_0(x)$, where $v_n(x)$ is given by (40).

3. The competitive best choice problems. Let us consider two person, zero-sum game approach to the classical best choice problem. Two employers, Player 1 and Player 2, are to view a group of $N$ applicants for a vacancies in their enterprises sequentially. We assume that the Player 1 has priority i.e. $\alpha_n = 1$. The solicited candidates are available with some probability only. It means that $\beta_n^1 = \beta_n^2 = p$. Using algorithm presented in Section 1 we can obtain the strategies for both players. Since the aim is the best applicant and $\alpha_n = 1$ then the Model A and Model B will be applied. We have got the following asymptotic equilibrium strategies.
Example 1. Model A.

\[(\epsilon^*, \delta^*) = \begin{cases} 
(s, s) & n \geq [Na], \\
(f, s) & [nb] \leq n < [Nc], \\
(s, f) & n < [Nb],
\end{cases}\]

where \( N \) is the number of candidates, \( a = p \frac{1}{1+p}, \ b = [p(2p - 1)]^{1/p} \).

Example 2. Model B.

\[(\epsilon^*, \delta^*) = \begin{cases} 
(s, s) & n \geq [Nc], \\
(f, s) & [nd] \leq n < [Nc], \\
(s, f) & \text{otherwise},
\end{cases}\]

where \( N \) is the number of candidates, \( c = (1 - (1 - p)^2)^{\frac{1}{(1-p)^2}}, \ b = p \frac{1}{1+p}[1 - (1 - p)^2]^{\frac{1}{(1-p)^2}} \).

This is only simple example to show that the considered models give the different optimal behaviour for the players.

REFERENCES


