Operations Research '92

Extended Abstracts of the 17th Symposium on Operations Research held at the Universität der Bundeswehr Hamburg at August 25-28, 1992

With 36 Figures

Physica-Verlag
A Springer-Verlag Company
Discrete-Time Markovian Jump Linear Systems

K. Szajowski
Institute of Mathematics, Technical University of Wroclaw
Wybrzeże Wyspiańskiego 27, 50-370 Wroclaw, Poland
W.L. de Koning
Faculty of Technical Mathematics and Informatics, Delft University of Technology
Mekelweg 4, 2628 CD Delft, Netherlands

The research is oriented to the control of discrete-time linear systems with randomly changing parameters which can be described by a finite-state Markov chain. The cost criterion is a quadratic form of the controls and states of the system. The criterion parameters also depend on the states of the Markov chain. Two models of observation of the Markov chain are adopted—delay for one step and undelay. It is shown that under appropriate mean square detectability and stability conditions the infinite horizon optimal control problem for the general case of Markovian jump linear quadratic systems has a unique solution when the control system is mean square stable. Necessary and sufficient conditions are given to determine if a system is mean square stable.

This model can be used to formulate systems that are subject to abrupt changes in their structure and components due to failure, repairs, changing subsystem interconnections or system environment. This type of systems is sometimes called a "hybrid system". The stability and detectability properties for these systems are investigated.

Such systems were studied by Ji and Chizeck (1990). The case of i.i.d. parameters was considered by many authors. For references see Ji and Chizeck (1990) and de Koning (1982). Related results concerning the stability of such systems were obtained by Costa and Fragoso (1991). The control of such systems were considered recently by Blom (1991) and Yang and Bar-Shalom (1991). However, these results were based on other assumptions about the parameters or knowledge taken into account in the construction of the control.

Formulation of the problem

The paper deals with a discrete time linear system with Markovian jumps, modeled by

\[ x_{k+1} = A(r_k)x_k + B(r_k)u_k, \quad (1) \]

where \( k = 0, 1, \ldots, N, \ x_k \in \mathbb{R}^n, \ u_k \in \mathbb{R}^m, \ A \in \mathcal{M}_E^n, \ B \in \mathcal{M}^{mn}_E. \ \mathcal{M}^{mn}_E \) denotes the space of all real \( m \times n \) matrices with norm \( \|A\| = \sup_{|x|=1} |Ax| \), where \( A \in \mathcal{M}^{mn}_E \) and \( x \in \mathbb{R}^n. \)

We denote \( \mathcal{M}^{nn}_E \) by \( \mathcal{M}^n \) and let \( \mathcal{S}^n \subset \mathcal{M}^n \) be a space of the real symmetric matrices. The zero and identity element in \( \mathcal{M}^n \) are \( O \) and \( I \), respectively. The set of all non-negative definite matrices of \( \mathcal{S}^n \) is denoted by \( \mathcal{K}^n \). Put \( \mathcal{I}E = \{1, 2, \ldots, s\} \). We will denote by \( \mathcal{M}^{mn}_E \).
the set of functions defined on $\mathcal{E}$ with values in $\mathcal{M}_{mn}$. Denote $\mathcal{I}$ the element of $\mathcal{M}_{n}$ such that $\mathcal{I}(i) = 1$ and let $\Theta \in \mathcal{M}_{\mathcal{E}}$ be such that $\Theta(i) = O$. For $f \in \mathcal{M}_{\mathcal{E}}^{mn}$ we define norm $\|f\| = \max_{r \in \mathcal{E}}(\|f(r)\|)$. $N$ can be finite or infinite. It is assumed that $x_0$ is given and $\{r_k\}_{k=0}^N$ is a homogenous Markov chain defined on a fixed probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathcal{E}$. Let $(p_i)_{i \in \mathcal{E}}$ and $(p_{ij})_{i,j \in \mathcal{E}}$ denote the initial and transition probabilities for this Markov chain, respectively. Let $\bar{u} = (u_0, u_1, \ldots, u_{N-1})$. In the finite horizon case, system (1) is considered with the cost criterion

$$J_N(\bar{u}, x_0) = \mathbb{E}\left\{ \sum_{i=0}^{N-1} [x_i^TQ(r_i)x_i + u_i^TR(r_i)u_i] + x_N^TH(r_N)x_N|x_0 \right\} \quad (2)$$

and in the infinite horizon case with the cost criterion

$$J(\bar{u}, x_0) = \mathbb{E}\left\{ \sum_{i=0}^{\infty} x_i^TQ(r_i)x_i + u_i^TR(r_i)u_i|x_0 \right\} \quad (3)$$

where $Q, H \in \mathcal{K}_F, \quad R \in \mathcal{K}_E$.

Denote $r^k = (r_0, r_1, \ldots, r_k)$ and $z^k = (x_0, u_0, \ldots, x_{k-1}, u_{k-1}, x_k)$. The different classes of admissible controls can be considered for system (1) with criterion (2) or (3). We focus our attention on two different classes of strategies:

(D0) $u_i = g_i(z^i, r^i)$ - the control at moment $i$ is based on information about the states, controls and states of the Markov chain up to moment $i$;

(D1) $u_i = g_i(z^i, r^{i-1})$ - uses the same information about the states and controls of the system as in (D0) but there is a one step delay in the observation of the state of the Markov chain.

We consider the optimal control problem for the classes of strategies (D0) and (D1). For i.i.d. random variables $\{r_n\}_{n=0}^N$, the knowledge of $r_0, \ldots, r_{k-1}$ has no influence for the posterior distribution of $r_k$. Taking this into account, the results of the paper for the class of control policies (D1) are a generalization of the considerations in De Koning (1982).

**Finite horizon optimal control**

Let us consider the case of controls (D1) i.e. based on the observation of states and for one-step-delayed observations of the Markov chain. We assume $P(r_{i-1} = 1) = 1$ and $P(r_0 = i|r_{i-1} = j) = p_{ij}$, $i, j \in \mathcal{E}$. For $A \in \mathcal{M}_{\mathcal{E}}^{mn}, B \in \mathcal{M}_{\mathcal{E}}$ adopt the following convention: $A(r)B(r) = [AB](r)$. The point of the following lemma is in the preliminary calculation of criterion (2) for the given controls. Denote $\bar{X}(r) = E[X(r_1)|r_0 = r]$ for $X \in \mathcal{S}_E$.

The cost criterion can be retyped in the equivalent form

$$J_N(\bar{u}, x_0) = \mathbb{E}\left\{ \sum_{i=0}^{N-1} [x_i^T\bar{Q}(r_{i-1})x_i + u_i^T\bar{R}(r_{i-1})u_i] + x_N^T\bar{H}(r_{N-1})x_N|x_0 \right\} \quad (4)$$
\[
J_{n,N}(\bar{u}, x^n, r^{n-1}) = E\left\{ \sum_{i=n}^{N-1} [x_i^T \tilde{Q}(r_{i-1}) x_i + u_i^T \tilde{R}(r_{i-1}) u_i] + x_N^T \tilde{H}(r_{N-1}) x_N | x^n, r^{n-1} \right\}.
\]

Let \( \Psi \in \mathcal{M}_{E \times E}^n \) and \( X \in \mathcal{M}_E^n \). Denote \( \overline{\Psi^T X \Psi}(r) = E[\Psi^T(r_0, r_1) X(r_1) \Psi(r_0, r_1) | r_0 = r] \).

**Lemma 1** Let us consider system (1) with cost criterion (4). Suppose

\[
u_i = -L(r_{i-1}) x_i,
\]

where: \( L \in \mathcal{M}_E^{m n} \), then for every \( x_0 \)

\[
J_N(\bar{u}, x_0) = x_0^T G_L^N \tilde{H}(r_{-1}) x_0,
\]

where: \( G_L : S_E^m \rightarrow S_E^n \) is defined by

\[
G_L X(r) = H_L X(r) + \tilde{Q}_L(r)
\]

where

\[
H_L X(r) = \overline{\Psi_L^T X \Psi_L}(r),
\]

\[
\tilde{Q}_L(r) = \bar{Q}(r) + [L^T \tilde{R} L](r) \text{ and } \Psi_L(r, s) = A(r) - B(r) L(s).
\]

By the dynamic programming principle for the systems with the Markovian parameters (see Szajowski & de Koning (1992)) we get

**Proposition 1** Let the system be described by (1) with the cost criterion given by (2). The solution of the finite horizon optimal control problem with a delay in the observation of Markov parameter \( r_n \) for one step is given by

\[
u_n^* = \begin{cases} 0, & n = 0, 1, \ldots, N - 1 \\ -L \bar{G}^{N-n-1} \tilde{H}(r_{n-1}) x_n & \end{cases}
\]

for \( n = 0, 1, \ldots, N - 1 \) and

\[
J_N(\bar{u}^*) = x_0^T \bar{G}^N \tilde{H}(r_{-1}) x_0,
\]

where \( \bar{G} X(r) = \bar{L} X X(r) \),

\[
\bar{L} X(r) = ((B^T X B + \bar{R})^+ B^T X A)(r).
\]

The open form of operator \( \bar{G} \) is as follows

\[
\bar{G} X(r) = \bar{Q}(r) + \overline{A^T X A}(r)
\]

\[
-[(A^T X B (B^T X B + \bar{R})^+ B^T X B)](r)
\]
Infinite horizon optimal control

The properties of operator $\mathcal{G}_*$ and $\mathcal{G}_L$ show that the finite horizon optimal control problem is an approximation for the infinite horizon control problem (see de Koning (1982)). We get

**Theorem 1** If system (1) is ms-stabilizable by a control dependent on the delayed observation of the Markov chain, then $S(r) = \lim_{N \to \infty} \mathcal{G}_*^N \Theta$ exists and $S$ is the minimal solution in $\mathcal{K}_{\mathcal{E}}^N$ of the equation $S = \mathcal{G}_* S$.

and

**Theorem 2** Let $S = \lim_{N \to \infty} \mathcal{G}_*^N \Theta$ exist and $u_i = -\mathcal{L}(r_{i-1})x_i$ for every $i \in \mathbb{N}$. We have

$$J(\bar{u}) = x_0^T S x_0.$$ 

We can also obtain the stability condition for the optimal control.

**Theorem 3** Let $S = \lim_{N \to \infty} \mathcal{G}_*^N \Theta$ exist. If $\bar{R}(r)$ is positive definite for every $r$, $(A, C)$ is ms-detectable or $\bar{Q}(r)$ is positive definite then system (7) with $\Psi(r, s) = \Psi_{L_S}(r, s)$ is stable and $S(r)$ it is the unique non-negative definite solution of the equation $S = \mathcal{G}_* S$.

The equivalent conditions and theorem can be stated for the system without a delay of the observation of the Markov chain. To this end, operator $\mathcal{G}_*$ should be replaced by $\mathcal{D}_*$ and $\bar{R}, \bar{Q}$ by $R, Q$, respectively.

On the basis of the solution of the infinite horizon optimal control problem one can state the criterion of stabilizability. Let us consider the stability of the close loop control system

$$x_{k+1} = \Psi(r_{k-1}, r_k)x_k$$  \hspace{1cm} (7)

for $k = 0, 1, 2, \ldots, N - 1$. System (7) can be obtained from (1) when we assume controls (5).

**Theorem 4** System (1) is ms-stabilizable by controls based on the delayed observation of the Markov chain if and only if sequence $\{K_n(r)\}$, obtained from the following recursive equation

$$K_n = I + A^T \bar{K}_{n-1} A(r) - A^T \bar{K}_{n-1} B (I + B^T \bar{K}_{n-1} B)^+ B^T \bar{K}_{n-1} A(r)$$  \hspace{1cm} (8)

with initial condition $K_0(r) = 0$ for every $r \in \mathcal{E}$, converges to a set of symmetric, non-negative definite matrices as $n$ tends to infinity.
References


