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On some selection problem

Abstract. Some optimal stopping problem is considered. N i.i.d. random variables with a known continuous distribution are observed sequentially. The number of observations N is random and the observer knows the distribution of N. The objective is to maximize the probability of selecting the largest or the second largest when observation is perfect, one choice can be made and neither recall nor uncertainty of selection is allowed. The special case with N having the geometric distribution is worked out.

Key words and phrases: optimal stopping, best choice problem

AMS (1980) Subject Classification: Primary 62L15, Secondary 60G40

1. Introduction.

The following full-information best choice problem was studied by Gilbert and Mosteller (1966). A known number, N, of i.i.d random variables \(X_1, X_2, \ldots, X_N\) from a known continuous distribution \(F\) are observed sequentially. The objective is to maximize the probability of choosing the largest. After \(X_n\) is observed it must be chosen (and the observation is terminated) or rejected (and the observation is continued). Neither recall nor uncertainty of selection is allowed and one choice must be made.

This problem for a finite number of observations was solved based on a heuristic argument by Gilbert and Mosteller (1966). The so-called monotone case was obtained. The optimal strategy is to accept (if possible) the first observation \(X_n\) which is largest one
so far and exceeds $x_n$, where the sequence of the optimal decision labels $x_1, x_2, \ldots, x_N$ is non-increasing.

The full-information best choice problem when the number of observation $N$ is random was considered by Porosiński (1987). A class of distribution of $N$ for which the monotone case occurs was characterized and the solution for this case was given.

In this paper the full-information best choice problem with one choice is considered when both the largest and the second largest are counted as wins. The number of observations $N$ is allowed to be a random variable with a known distribution.

This problem is reduced to the classical optimal stopping problem for some Markov chain. The special case when $N$ has geometric distribution is examined in details.

2. Preliminaries.

Assume that

(1) $X_1, X_2, X_3, \ldots$ is a sequence of iid random variables with a continuous distribution function $F$, defined on the probability space $(\Omega, \mathcal{F}, P)$.

(2) the number of observations $N$ is a random variable independent of $(X_n)_{n=1}^{\infty}$ with a known distribution $P(N=n) = p_n, n=0, 1, 2, \ldots, \sum_{n=0}^{\infty} p_n = 1$.

Let $\mathcal{F}$ be the set of all Markov moments with respect to the family of $\sigma$-fields $(\mathcal{F}_n)_{n=1}^{\infty}$, where $\mathcal{F}_n = \sigma(X_1, \ldots, X_n, 1_{\{N=0\}}, \ldots, 1_{\{N=n-1\}})$ ($1_\lambda$ denotes the indicator function of the event $\lambda$).

Let $\xi_n$ and $\eta_n$ denote the largest and the second largest value of the sequence $X_1, \ldots, X_n$, respectively. Consider the following problem:

(P) Find a stopping time $\tau \in \mathcal{F}$ such that

$P(\tau \leq N, X_\tau = \xi_N \text{ or } X_\tau = \eta_N) = \sup_{\tau \in \mathcal{F}} P(\tau \leq N, X_\tau = \xi_N \text{ or } X_\tau = \eta_N).$

Since $F$ is known and continuous, without loss of generality may be additionally assumed that $X_n$ has the uniform distribution on the interval $[0, 1]$, $n \in \{1, 2, \ldots\}$. Denote
\[ Z_n = P(N \geq n, X_n = \xi_n \text{ or } X_n = \eta_n \mid \mathcal{F}_n) \]

\[ = I(X_n = \xi_n \text{ or } X_n = \eta_n) \sum_{m=n}^{\infty} P(N = m, X_n \text{ is largest or second largest from } X_n, \ldots, X_m \mid \mathcal{F}_n) \]

\[ = I(X_n = \xi_n \text{ or } X_n = \eta_n) W_n, \]

where

\[ W_n = \sum_{m=n}^{\infty} \left( \frac{p_n}{\pi_n} \right)^m \left[ I(X_n = \xi_n) \left( \frac{\gamma_n^{m-n}}{\gamma_n^{m-n} + \gamma_n^{m-n-1}(1-x_n)} \right) I(X_n = \eta_n) \gamma_n^{m-n} \right], \]

\[ \pi_n = P(N \geq n) = \sum_{k=n}^{\infty} p_k, \]

and \( Z_n = 0 \) for \( n > M \) where \( M = \inf(n: \pi_n > 0) \) (\( M = +\infty \) if \( N \) is unbounded). Hence

\[ EZ = P(\tau \leq N, X_\tau = \xi_n \text{ or } X_\tau = \eta_n). \]

It suffices to consider Markov moments belonging to the set of candidates

\[ \mathcal{J}_\infty = \{ \tau \in \mathcal{J} : \tau = n \text{ or } X_\tau = \xi_n \text{ or } X_\tau = \eta_n, n \in \mathbb{N} \}. \]

Now, let

\[ \tau_k = \begin{cases} 1 & \text{if } N > 1, \\ +\infty & \text{if } N = 0, \end{cases} \]

\[ \tau_{k+1} = \inf \{ n : n > \tau_k, n \leq N, X_n = \eta_{\tau_k} \}, k \in \mathbb{N}, \]

and let the random variable \( R_n \) indicates a range of \( n \)th observation

\[ R_n = \begin{cases} 1 & \text{if } X_n = \xi_n, \\ 2 & \text{if } X_n = \eta_n, \\ 0 & \text{otherwise}. \end{cases} \]

Define, for \( k \in \mathbb{N} \),

\[ Y_k = \begin{cases} (\tau_k, \xi_{\tau_k}, \eta_{\tau_k}, R_{\tau_k}) & \text{if } \tau_k < +\infty, \\ \delta & \text{if } \tau_k = +\infty, \end{cases} \]

where \( \delta \) is a label for the final state.

\( Y = (Y_k)_{k=1}^{\infty} \) is a Markov chain with respect to \((\mathcal{Y}_k)_{k=1}^{\infty}\). The state space of this chain is
where

\[ K = \{1, \ldots, M\}, \]
\[ A = \{(ξ, η): 0 ≤ η ≤ ξ ≤ 1\}. \]

The transition function is

\[
p(n, r, s, i; m, [0, x], r, 1)
= P(τ_{k+1} = m, ξ_m ≤ x, η_m ≤ r, R_m = 1 | τ_k = n, ξ_n = r, η_n = s, R_n = i)
= \begin{cases}
(p_{m/n} s^{m-n-1}(x-r)) & \text{if } x ≤ r, \\
0 & \text{otherwise},
\end{cases}
\]

\[
p(n, r, s, i; m, r, [0, x], 2)
= P(τ_{k+1} = m, ξ_m = r, η_m ≤ x, R_m = 2 | τ_k = n, ξ_n = r, η_n = s, R_n = i)
= \begin{cases}
(p_{m/n} s^{m-n-1}(x-s)) & \text{if } s ≤ x ≤ r, \\
(p_{m/n} s^{m-n-1}(r-s)) & \text{if } x > r, \\
0 & \text{otherwise}
\end{cases}
\]

for \( m > n, i = 1, 2 \) (\( m \in K \)). The state \( δ \) is absorbing and the transition function for other states can be obtained in a similar way.

If, for any \( τ \in J_0 \), a Markov moment \( σ \) with respect to \((S_τ)_{k=1}^∞\) is defined as: \( σ = k \) on the set \( \{τ-k<+∞\}, k \in K \), and \( σ = +∞ \) on \( \{τ = +∞\} \), then

\[
Z_τ = \begin{cases}
W_{τ, σ} & \text{if } τ < +∞ \\
0 & \text{if } τ = +∞
\end{cases}
= f(Y_τ),
\]

where \( f(δ) = 0 \) (\( Y_∞ = δ \) by definition) and

\[
f(n, x, y, i)
= \begin{cases}
\sum_{m=n}^∞ (p_{m/n} s^{m-n}) & \text{if } 0 ≤ y ≤ x ≤ 1, i = 2, \\
\sum_{m=n}^∞ (p_{m/n} (x^{m-n}+(m-n)x^{m-n-1}(1-x))) & \text{if } 0 ≤ y ≤ x ≤ 1, i = 1, \\
0 & \text{if } x < y, i = 1, 2.
\end{cases}
\]

In this way the initial Problem (P) is led to the problem of optimal stopping of the Markov chain \( Y \) given by (3) with the reward function \( f \) given by (4).
3. Reduction.

Now

$$s(n,x,y,i) = \sup_{\tau \in \mathbb{N}} E_{n,x,y,i} \int f(\tau, x_\tau, y_\tau, R_\tau)$$

should be calculated, where $E_{n,x,y,i}$ denotes the expectation with respect to $P_{n,x,y,i}(\cdot) \cdot \mathcal{P}(n,x,y,i; \cdot)$ and an optimal $\tau$ ought to be displayed. It is known (cf Shiryaev (1969)) that $s(\epsilon)$ for $\epsilon \in E$ satisfies

$$(5) \quad s(\epsilon) = \max(f(\epsilon), \mathcal{P}(\epsilon)),$$

where

$$\Phi(\epsilon) = \int_E h(a) d\mathcal{P}_\epsilon(a)$$

for a bounded function $h : E \to \mathbb{R}$. So $\Phi(\delta) = 0$ and from (4)

$$\Phi(n,x,y,i) = \sum_{n=1}^{\infty} \left\{ \int_0^X h(m,x,s,2) (n_m/n_n) y_{m-n-1} ds + \int_1^X h(m,r,x,1) (n_m/n_n) y_{m-n-1} dr \right\}$$

for $i = 1, 2$.

It is well known that the Markov moment

$$\tau_0 = \inf(n \in \mathbb{N} : Y_n \in \Delta),$$

where

$$\Delta = \{ \epsilon \in E : s(\epsilon) = f(\epsilon) \}$$

($\Delta$ is called the stopping set), is optimal if $\tau + \omega$ almost surely (a.s.). In order to calculate an optimal strategy in Problem (P), it suffices to investigate the stopping set, because the chain $Y$ attains a.s. the state $\delta$ and $\delta \in \Delta$.

Since $f(n,x,y,1) \leq f(n,x,y,2)$ for every $(x,y) \in \mathcal{A}$ and $\mathcal{P}(n,x,y,i)$ does not depend on $i$, there exist subsets $A_n \subseteq A_n$, $n \in \mathbb{N}$, such that

$$\Delta = \bigcup_{n=1}^{\infty} (n \times \{1\}) \cup A_n \times \{2\} \cup \{ \delta \}.$$

In the general case, it is very difficult to determine analytically the stopping set. Nevertheless in a natural case, which is considered below, a solution has a simple form.
4. The geometric case.

Let \( p_k = pq^k, k = 0, 1, \ldots, 0 < p, p + q = 1 \). Results for this case are summarized in the following theorem.

Theorem. Assume that \( N \) has the geometric distribution with parameter \( p \).

(a) If \( p < \alpha \), where \( \alpha = 0.30171 \) is a root of the equation \( 2u - 2\ln u - 3 = 0 \) in (0, 1) (or, equivalently, if \( 2p - 2\ln p - 3 > 0 \)), then

\[
\tau^* = \inf \{ n : X_n \geq (1 - \alpha^{-1})p/q \}
\]

is the solution of Problem (P) and the probability that, using this stopping rule, the largest or the second largest \( X \) is obtained is equal to \( \alpha(2 - \alpha) = 0.51239 \).

(b) If \( p \geq \alpha \), then \( \tau^* = 1 \) is a solution of (P) and the above probability is \( p(p - 1 - 2\ln p) \).

Proof. For the geometric distribution it is easy to obtain

\[
f(n, x, y, 1) = 2(\frac{p}{(1-qx)} - \frac{p}{(1-qx)})^2,
\]

(6) \( f(n, x, y, 2) = \frac{p}{(1-qy)}, \)

\[
Pf(n, x, y, i) = \left( \frac{p}{(1-qy)} \right) \left( \frac{(p/(1-qx)) - 1 - \ln(p/(1-qx))(p/(1-qy))}{(p/(1-qy))} \right)
\]

\( i = 1, 2, (x, y) \in \Lambda, n \in \mathbb{N} \).

Notice that all these functions do not depend on \( n \). This implies that \( s(n, x, y, i) \) is independent of \( n \). Thus sets \( \Lambda_n^1, \Lambda_n^2 \) do not depend on \( n \) as well. In the sequel, \( n \) will be omitted in \( f, Pf, s \) and \( \Lambda_n^i \).

From the general theory of optimal stopping it is well known that

\[
s(x, y, i) = \lim_{k \to \infty} Q^k f(x, y, i),
\]

where

\[
Qf(x, y, i) = \max(f(x, y, i), Pf(x, y, i))
\]

and the operator \( P \) is defined as
\[ Ph(x,y,i) = \sum_{n=1}^{\infty} \left( \int_{y}^{x} h(x,s,2)q^{m}y^{m-1}ds + \int_{x}^{1} h(s,x,1)q^{m}y^{m-1}ds \right) \]
\[ = \left( q/(1-\eta y) \right) \left( \int_{y}^{x} h(x,s,2)ds + \int_{x}^{1} h(s,x,1)ds \right). \]

The form of \( Qf(x,y,i) \) is
\[ Qf(x,y,i) = \begin{cases} f(x,y,i) & \text{if } (x,y) \in B^i, \\ Pf(x,y,i) & \text{if } (x,y) \in A - B^i, \end{cases} \]
for some sets \( B^i, B^i \subseteq A \) and \( B^i \subseteq B^i \) because \( Ph(x,y,i) \) does not depend on \( i \). Let
\[ u = p/(1-\eta x), \ v = p/(1-\eta y). \]
Then functions (6) can be written as functions of \( u \) and \( v \)
\[ f(u,v,1) = 2u-u^2, \]
\[ f(u,v,2) = v, \]
\[ Pf(u,v,i) = v(u-1-lnuv), \ i=1,2, \]
and the set \( A \) is transformed into \( \{(u,v): p \leq v \leq u \leq 1\} \).

Equations of curves:
- \( OA: v = \exp(\nu-2)/u \)
- \( BC: 2u-u^2+v(1-u-lnuv)=0 \)

Fig. 1
The set \( B_i \) is defined by the inequality \( f(x,y,i) \geq Pf(x,y,i) \) or, equivalently, by \( f(u,v,i) \geq Pf(u,v,i) \) with conditions \( p \leq v \leq u \leq 1 \). These inequalities, considered in a triangle \( 0 \leq v \leq u \leq 1 \), are fulfilled in the sets

\[
C^i = \{(u,v): 2u - u^2 + v(1 - u + ln(u)) \geq 0\},
\]

\[
C^2 = \{(u,v): v \geq \exp(u-2)/u\},
\]

respectively (cf Fig.1). Since all functions (8) are independent of \( p \), the sets \( B_i \) in the coordinates \( u,v \) are

\[
B_i = C^i \cap \{(u,v): p \leq v \leq u \leq 1\}
\]

and in the coordinates \( x,y \) their forms are similar, because the transformation (7) keeps monotonicity.

In order to obtain \( s(x,y,i) \) successive iterations of \( Qf(x,y,i) \) should be calculated. Since \( Q^2 f(x,y,i) = \max(f(x,y,i), \quad P,Qf(x,y,i)) \) and \( Pf(x,y,i) \geq f(x,y,i) \) for \( (x,y) \in A-B_i \),

\[
P,Qf(x,y,i) = (q/(1-qy)) \left[ \int_x^X Qf(x,s,2) ds + \int_x^1 Qf(s,x,1) ds \right]
\]

\[
= Pf(x,y,i) \quad \text{if} \quad (x,y) \in B^2
\]

and \( P,Qf(x,y,i) > Pf(x,y,i) \) if \( (x,y) \in A-B^2 \). Thus

\[
Q^2 f(x,y,2) = \begin{cases} f(x,y,2) & \text{if} \quad (x,y) \in B^2 \\ Pf(x,y,2) & \text{if} \quad (x,y) \in A-B^2 \end{cases},
\]

\[
Q^2 f(x,y,1) = \begin{cases} f(x,y,1) & \text{if} \quad (x,y) \in B^1 \\ P,Qf(x,y,1) & \text{if} \quad (x,y) \in A-B^1 \end{cases},
\]

for some subset \( B^1 \) such that \( B^1 \leq B^2 \leq A-B^2 \).

As a consequence of induction, it is easy to obtain that

\[
s(x,y,2) = \begin{cases} f(x,y,2) & \text{if} \quad (x,y) \in B^2 \\ Pf(x,y,2) & \text{if} \quad (x,y) \in A-B^2 \end{cases},
\]

and there exists a sequence \((B_i^n)_{n=1}^{\infty}\) such that

\[
Q^n f(x,y,1) = \begin{cases} f(x,y,1) & \text{if} \quad (x,y) \in B^1 \\ P,Q^{n-1} f(x,y,1) > f(x,y,1) & \text{if} \quad (x,y) \in A-B^1 \end{cases},
\]

where \( B^2 \leq B^1 \leq B^0 \), \( n \geq 2 \).
So there exists $A^f = \lim_{n \to \infty} B^f_n$ and the stopping set has the form

$$\Delta = \bigcup_{x \in A^f} \left( x \times \{1\} \cup A^2 \times \{2\} \right) \cup \{0\},$$

where $A^2 - B^2$.

In order to examine the set $A^f$ denote by $h(x,y)$ the probability that the stopping set $\Delta$ will be reached in the future, if the present state is $(x,y,i)$ (the function $h(x,y)$ does not depend on the fact, whether the largest or the second largest is observed). For $(x,y) \in A-B^2$, if the next candidate having value $r$ fulfills the inequality $y < r < x$, it does not attain the set $B^2$ for $y < r < y_o - \min(x, y(x))$, where the function

$$y(x) = \left[ 1 - p^2 \exp(p/(1-qx) - 2)/(1-qx) \right] / q$$

describes a boundary of the set $B^2$. If $r > x$ then the new state $(r,x,1)$ does not depend on $y$. Thus

$$h(x,y) = \left( q/(1-qy) \right) \left[ \int_y^\infty h(x,r) \, dr + \text{some func. indep. of } y \right]$$

and the partial derivative of $h$ with respect to $y$ is equal to 0. So $h(x,y)$ is independent of $y$. Since $f(x,y,1)$ is also independent of $y$, if $(x,y) \in A^f$ then $(x,r) \in A^f$ for every $0 < r < x$. The set $A^f$ has the form

$$A^f = \{(x,y) \in A : x \geq L\},$$

where $\min\{1,x_2\} \geq L \geq \max\{0,x_1\}$ and $x_1, x_2$ are solutions of the equations

$$3 - 2p/(1-qx) + 2\ln(p/(1-qx)) = 0,$$

$$\left( p/(1-qx) \right)^2 - \exp(p/(1-qx) - 2) = 0,$$

respectively (cf forms of $B^f$ and $B^2$).

As the result of above properties, the optimal strategy allows to stop observation only in such a moment, when an observed candidate is the largest one so far and exceeds some value $L$ (independent of the second largest to that moment). So the optimal strategy can be obtained by maximization of the probability of attaining of $A^f$.

Let the event in which the objective is achieved will be called a win and its probability will be called a probability of
winning. Then

\[ P(\text{stop at the moment } k \& \text{ win } | \ N=n) \]
\[ = P(X_1 \leq L, \ldots, X_{k-1} \leq L, X_k > L, \ \text{max}(X_{k+1}, \ldots, X_n) \leq X_k \text{ or exactly one from } X_{k+1}, \ldots, X_n \text{ is greater than } X_k) \]
\[ = L^{k-1} \int_L^{1-n-k+1} (n-k) x^{n-k-1} (1-x) \, dx_k \]
\[ = L^{k-1} \left[ \frac{1-L^{n-k}-(n-k-1)(1-L^{n-k+1})}{(n-k+1)} \right] \]
and

\[ P(\text{win}) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} P(\text{stop at } k \& \text{ win } | \ N=n) \, P(N=n) \]
\[ = \sum_{n=1}^{\infty} pq^n L^{n-1} \left[ 2(L^{-r-1})/r - L^{-1}(1-L) \right] \]
\[ = \sum_{r=1}^{\infty} p \left[ 2(L^{-r-1})/r - L^{-1}(1-L) \right] \sum_{n=r}^{\infty} (qL)^n \]
\[ = (p/(1-qL)) \left[ p/(1-qL) - 2/n (p/(1-qL)) - 1 \right]. \]

Let \( p/(1-qL) = u \). The function \( f(u) = u(u-2lnu-1) \) in \((0,1)\), connected with \( P(\text{win}) \), has a unique local extremum (maximum) in \((0,1)\) at the point \( \alpha \) for which the derivative \( f'(u) = 2u-2lnu-3 \) is equal to zero. So, since \( u \in [p, 1] \), the function \( f(u) \) attains its maximum at \( \alpha \) if \( p < \alpha \) (or equivalently if \( f'(p) > 0 \)) or at \( p \) if \( p \geq \alpha \). Thus \( L^* = (1-\alpha^{-1})p/q \) and \( P(\text{win}) = f(\alpha) = \alpha(2-\alpha) \) if \( p < \alpha \) or \( L^* = 0 \) and \( P(\text{win}) = f(p) \) if \( p \geq \alpha \). The theorem is proved.

5. Remarks.

1. The full-information best choice problems have been solved, as far as the authors know, only for choosing the largest by accepting exactly once when the number of observations \( N \) is fixed or random (cf. papers of Gilbert and Mosteller (1966) and Poresiński (1987)) and for choosing the largest by accepting in twice when \( N \) is fixed (Tamaki (1980)).

2. In the geometric case it is interesting and quite unexpected that the probability of winning in all natural situations (i.e. when \( p \) is small) is constant independent of \( p \) (see Theorem). The optimal strategy does not depend on the number of preceding
observations. The same property is fulfilled for the optimal strategy in selecting the largest with $N$ geometrically distributed (Porosiński (1987)). This interesting fact is supposed to be a consequence of the memoryless property of the geometric distribution.

References.


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