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Analysis of the Structure and the Reliability of the Multistage Technical Systems

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Abstract. The paper is concerned with introducing new and improving known methods of description and reliability investigation of compound technical systems. The reliability structure is described as a family of convex sets or as a matroid or as some graph. Based on these description a new look at the reliability of the compound systems is proposed. Relations this approach to the reliability of the systems and random graphs are given.

Keywords: clutter, module, random graph, reliability of the system, simulation, structure of the system

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1. Introduction.

A compound system is able to realize different functions. The degree of efficiency of the system (the state of the system) depends on the quality and sort of functions which can be realized by the system. It is assumed that the state of the system is determined solely from a knowledge of which components are able to work. This look at the system is wider then those known in literature, as for example in nice Barlow & Proshan's monographic book [1].

To discuss such point of view and related computing procedures we need to define precisely the reliability structure of such systems. This is the aim of the paper.

In Section 2 the concept of description of the structure by a function \( f \) (called in future the reliability structure) defined on subsets of the set \( X \) of the elements of the system under consideration is given. It is generalization of the structure function introduced by Barlow & Proshan [1]. It is shown that the system can be also described by a family of convex sets. Next, the relations between the structure \( f \) and the clutter of a finite set \( X \) is considered in Section 3. Especially important for these considerations is the concept of module (see [1]) which in engineering term refers to a "package" of components which can be replaced as a whole. Connections of the module and the structure \( f \) are given in Section 4. Section 5 contains description of the graph representation for some systems. Based on these definitions,
Now we consider two properties: exchange property

(E) \[ x, y \notin f(A), x \in f(A \cup \{y\}) \Rightarrow y \in f(A \cup \{x\}). \]

and anti-exchange property

(AE) \[ x \neq y, x, y \notin f(A), x \in f(A \cup \{y\}) \Rightarrow y \notin f(A \cup \{x\}). \]

If the system \((X, f)\) fulfills additionally the condition (E) then \((X, f)\) is a matroid (see [12]) and if the system fulfills (AE) then \((X, f)\) is an anti-matroid (see [2]).

If the elements of \(A\) fail and \(B = f(A)\) we say that the system \((X, f)\) is in the state \(B\). If \(A = \emptyset\) (and obviously \(B = \emptyset\)) we say that the system is completely efficient. If, with some \(A\), \(f(A) = X\), then we say that the system is completely failed. If \(B \neq \emptyset\) and \(B = f(A) \subset X\), then we say that the system is partially efficient. Therefore the state \(\emptyset\) in Barlow & Proshan’s terminology denotes the state of complete failure and \(1\) denotes the state of complete or partial efficiency.

There are at least two possible interpretations of axioms \((s1) - (s2)\). The first one is the following: the equality \(f(A) = B\) means that if the elements of \(A\) fail then all elements of \(B\) cannot work. The second interpretation is: if the elements of \(A\) fail and we have to release them, then we get a possibility to release all \(B = f(A)\) elements, where the elements of \(B \setminus A\) may be replace preventively (see [4]). In the next sections of our paper we will use only the first of the above interpretation.

3. Clutter representation.

A nonempty family \(P\) of subsets on a finite set \(X\) is called a clutter, if \(P \neq \{\emptyset\}\) and no elements of \(P\) are properly contained in another element of \(P\).

The structure \(f\) can be defined as follows

\[
(1) \quad f(A) = \{x \in X : \forall_{B \in P} x \in B \Rightarrow A \cap B \neq \emptyset\}.
\]

**Proposition 1.** The function \(f\) defined by (1) is a reliability structure.

**Proof:** The function \(f\) fulfills the conditions \((s1)-(s4)\), therefore it is a reliability structure.\]

If for a given structure \(f\) there exists a clutter \(P\) such that (1) is fulfilled, we say that the system \((X, f)\) has a clutter representation \((X, P)\) or shortly that \(f\) has clutter representation \(P\).

Barlow & Proshan introduced (see [1]) a family of paths defined as a minimal set of working elements such that

\[
(2) \quad \varphi(x_1, ..., x_n) = 1.
\]

It is obvious that the set of paths forms a clutter. Therefore one can obtain the structure function \(f\) from \(\varphi\) via the formula (1). Such a way of getting the structure \(f\) we call a natural extension of \(\varphi\) to \(f\).

Therefore, we have a fairly tautological
the possibilities of determining the probability, that the system is in a given state, is discussed in Section 6 and 7. A simple example is given in Section 8. The application of these results to analysis of the reliability of mechanical systems was given by the authors in [8].

2. Foundations.

Let \( X \) be a finite set of elements, \( X = \{x_1, \ldots, x_n\} \) and \( f : 2^X \rightarrow 2^X \) a function satisfying the relations:

\[
\begin{align*}
(s1) & \quad f(\emptyset) = \emptyset, \\
(s2) & \quad A \subseteq f(A), \\
(s3) & \quad f(A) \cup f(B) \subseteq f(A \cup B), \\
(s4) & \quad f(f(A)) = f(A).
\end{align*}
\]

The pair \((X, f)\) will be called the system and the function \( f \) the (reliability) structure.

The structure \( f \) is a generalization of the structure function \( \varphi : 2^X \rightarrow \{0, 1\} \) introduced by Barlow & Proschan [1]. It was assumed that \( \varphi \) is nondecreasing and all elements are relevant, i.e. there are no elements such that

\[
\varphi(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) = \varphi(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)
\]

for all \((x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\). In this paper we do not make such assumption.

By a convex set \( A \) we mean a subset such that \( f(A) = A \). Let \( \mathcal{B} \) denotes a family of convex sets:

\[
\mathcal{B} = \{A : f(A) = A\}.
\]

It is well known (see [2]) that \( \mathcal{B} \) is a family of convex sets if and only if

\[
\begin{align*}
(c1) & \quad \emptyset \in \mathcal{B}, \quad X \in \mathcal{B}, \\
(c2) & \quad A, B \in \mathcal{B} \Rightarrow A \cap B \in \mathcal{B}.
\end{align*}
\]

Moreover we have

\[
(s) \quad f(A) = \bigcap_{A \subseteq B \in \mathcal{B}} B.
\]

Therefore by structure we can mean also a pair \((X, \mathcal{B})\), where \( \mathcal{B} \) fulfills the conditions \((c1) - (c2)\).

Note that the condition \((s3)\) can be replaced by

\[
(s3') \quad A \subseteq B \Rightarrow f(A) \subseteq f(B).
\]
Proposition 2. A system \((X, f)\) has a clutter representation if and only if it is a natural extension of \(\varphi\), where

\[
\varphi(u_1, ..., u_n) = \begin{cases} 
1, & \text{for } f(A) \neq X, \\
0, & \text{for } f(A) = X,
\end{cases}
\]

and \(A = \{x_i \in X : u_i = u(x_i) = 1\}\). The above representation is unique.

If \(f\) has a clutter representation and consequently \(f\) is a natural extension of some \(\varphi\), then we say that \(\varphi\) is a binary representation of \(f\). All those relationships between \(\varphi\), \(\mathcal{P}\) and \(f\) are explained on the diagram below:

\[
\begin{array}{c}
\text{clutter repres.} \\
\varphi \xleftarrow{(2)} \mathcal{P} \xrightarrow{(1)} f \\
\text{natural extension: (2) and (1)} \\
\text{binary representation: (3)}
\end{array}
\]

Fig. 1

For a given clutter \(\mathcal{P}\) we define the blocker \(\overline{\mathcal{P}}\) as

\[
\overline{\mathcal{P}} = \{A \subseteq X; \forall B \in \mathcal{P} A \cap B \neq \emptyset, A' \subset A \in \mathcal{P} \Rightarrow A' \not\in \mathcal{P}\}
\]

i.e. \(\mathcal{P}\) is a family of minimal sets such that each one has a nonempty intersection with every member of \(\mathcal{P}\). It is well-known that each blocker is a clutter and \(\overline{\mathcal{P}} = \mathcal{P}\).

Considering a reliability system \((X, \varphi)\) and taking as a clutter a family of paths we obtain as a blocker the family of cuts (see [1]). Next, taking such a blocker as a family of paths we obtain a dual system \((X, \overline{\varphi})\) with its natural extension denoted by \((X, \overline{f})\). For a function \(\varphi\) the following formula holds:

\[
\overline{\varphi}(u_1, ..., u_n) = 1 - \varphi(1 - u_1, ..., 1 - u_n)
\]

but for our structures \(f\) and \(\overline{f}\) a similar relationship does not hold.

A particular example of a clutter is a family of cycles of some matroid (see [12]). If all members of a given clutter have the same number of elements, then in some cases the clutter may be a family of bases of matroids. In such a case we have the following result.
PROPOSITION 3. If the system has the clutter representation \((X, \mathcal{P})\) and \(\mathcal{P}\) forms a family of all bases in some matroid, then \(f(A) = \sigma(A)\) where \(\sigma\) is a span in a matroid, i.e. is the smallest submatroid containing \(A\).

Another example closely related to a matroid is a port, (see [11]) which arises from a matroid by choosing cycles containing a fixed element, say \(e\), and next, by removing \(e\) from chosen cycles. Note that if \(\mathcal{P}\) is a port then \(\mathcal{P}\) is also a port, which arises from a dual matroid with the same fixed element \(e\).

Finally note that any matroid of cycles is also a port (see [7]).


The concept of modules of coherent systems was introduced in [1]. In the system \((X, \varphi)\) the subsystem \((Y, \psi)\) is called a module if

\[
\varphi(x) = \eta(\psi(x|A) \oplus (x|X\setminus A))
\]

where \(x|A\) denotes the vector with elements \(x_i, e_i \in A\) and \(x \oplus y\) denotes a concatenation of \(x\) and \(y\), i.e. if \(x = (x_1, ..., x_m), y = (y_1, ..., y_n)\) then \(x \oplus y = (x_1, ..., x_m, y_1, ..., y_n)\). Function \(\eta\) is some structure function.

In the system \((X, f)\) we define a module \((Y, g)\) in the following way.

\[
\begin{align*}
(m1) & \quad Y \subseteq X, \\
(m2) & \quad g(A) = f(A) \cap Y \text{ for } A \subseteq Y, \\
(m3) & \quad f(A) = f(\Theta_Y(g(Y \cap A))) \cup (A \setminus Y) \cup g(Y \cap A), \text{ for } A \subseteq X,
\end{align*}
\]

where for \(B \subseteq Y:\)

\[
\Theta_Y(B) = \begin{cases} 
Y, & \text{for } B = Y, \\
\emptyset, & \text{for } B \neq Y.
\end{cases}
\]

The crucial role of the module can be resumed as follows. In practical reliability analysis, the procedure often followed is to compute first the reliability of each of the disjoint subsystems comprising a system, and then compute the overall system reliability from these subsystem reliability. The module is the precise definition of a "package" of components which can be replaced as a whole.

If \((X, f)\) has a clutter representation then we have the following result.

THEOREM 1. Let \((X, f)\) be a natural extension of \((X, \varphi)\). The \((Y, g)\) is a module of \((X, f)\) if and only if \((Y, \psi)\) is a module of \((X, \varphi)\), where \((Y, g)\) is a natural extension of \((X, \psi)\).

PROOF: The statement of Theorem 1 is a simple consequence of the following Lemma 1.

Let \(\mathcal{E}\) be family of subsets of \(X\). Denote \(\mathcal{E}(A) = \{B \in \mathcal{E} : A \cap B \neq \emptyset\}\).
LEMMA 1. A system \((Y, \psi)\) is a module of the system \((X, \varphi)\) if and only if

\[ P(Y) = \{ B \in P : B = (C \cap Y) \cup (D \setminus Y); C, D \in P(Y) \}, \]

where \(P\) is the clutter representation of \(\varphi\) and \(P(Y)\) is the clutter representation of \(\psi\).

In fact, the statement of Lemma 1 is equivalent to the definition of a module (see [10]).

The result of Theorem 1 gives us some algorithms to find the smallest module containing a given set (see [10]) and it allows us to decompose \((X, f)\) as well as \((X, \varphi)\) into modules.

5. Graph representation

The theory given in the above section has a combinatorial nature, the language of graph theory provides a natural description of the reliability structure in a lot of cases.

At first we consider a simple representation of closed sets. Let \(G = (X, R)\) be a directed graph on the set of vertices \(X\) and \(\Gamma(x) = \{ y : xRy \}\). If we denote

\[(cr) \quad B = B(G) = \{ A \subseteq X : \forall x \in X (x \in A \Rightarrow \Gamma(x) \subseteq A) \} \]

then we obtain the following result (see [6]):

THEOREM 2. The family \(B(G)\) defined by \((cr)\) fulfills \((c1) - (c2)\), therefore \(B(G)\) is a family of convex sets.

Let \(B\) be a family of convex sets and \(f\) be defined by \((s)\). Then a graph \(G\), such that \(B = B(G)\) is defined by \((cr)\), exists if and only if

\[ f(A) = \bigcup_{x \in A} f(\{x\}). \]

The above representation is helpful when at the same time only one break-down may occur.

Now we focus our attention on the clutter representation of a system. It seems that the most natural way to describe clutters in the reliability theory is by path in some graph. The elements of a system \((X, f)\) are edges or arcs of a graph.

Let \(G = (V, X)\) be a graph without loops, not obviously symmetric or directed. If all arcs (i.e. directed edges) form an acyclic graph and \(u\) and \(v\) are two fixed vertices, then arcs and edges in any path joining \(u\) and \(v\) form a set in a clutter.

Seymour in [5] gives necessary and sufficient conditions for a given clutter to be the collection of edge sets of the simple paths between \(u\) and \(v\) in symmetric graph.
In the theorem some difficult concepts from matroid theory are used and this result is not formulated in our paper.

In the above representation of clutters we take the edges and arcs of a graph to be elements of a system.

If a clutter is a family of bases of a matroid (see Section 3) and the matroid is graphical then the above problem may be considered as the problem of connectedness of a graph.

Now we consider another possibility, namely when elements of the system \((X, f)\) are vertices of a directed acyclic graph \(G = (X, R)\). In this case one can define a clutter as a family \(\mathcal{P}\) of all minimal sets of vertices \(A\) such that in any induced subgraph \(G_A, A \in \mathcal{P}\), any input vertex in \(G\) is joined with any output vertex in \(G\).

In particular case, when \(G = (X, R)\) has only one input and one output, a clutter \(\mathcal{P}\) is a set of all minimal vertex paths joining the terminal vertices.

Note, that if we consider not only a structure of a system, but also the probability that the elements be able to work, it leads us to random graphs and enables us to use that theory. A more detailed discussion will be provided in the next section.

6. The probabilistic point of view.

Reliability theory is concerned with determining the probability that the system, possibly consisting of many components, will function. From the above consideration we can see that it can be extended to determine the probability that a system realizes some specified functions. To make this approach more precise the description of reliability structure by the family of convex sets is chosen.

Let \((\Omega, \mathcal{F}, \Pr)\) be a fixed probability space. The behaviour of the system in time is modeled by a stochastic process \(\xi_t : \Omega \to \mathcal{B}\). The process \(\xi_t\) at time \(t\) is in the state \(A \in \mathcal{B}\) when the function realized by the subsystem consisting of the elements of \(A\) is not realized. Let \(\tau_A = \inf\{t : \xi_t = A\}\). The random moment \(\tau_A\) is the first moment when this subsystem stopped working. The reliability of the system to realize this function is defined as \(G_A(t) = \Pr(\tau_A > t)\).

The function \(G_A(t)\) can be determined analytically as follows. The elements of set \(A \in \mathcal{B}\) form the subsystem which works or is failed i.e. it is in two states. The reliability structure of it is described by the structure function \(\varphi : A \to \{0, 1\}\), fulfilling the condition of the structure function of the coherent system. This allows us to calculate the life time distribution \(G_A(t)\) of subsystem \(A\) based on the common distribution of life of the elements and the structure function \(\varphi\). For complicated system it is only a theoretical way. What can be done is the calculation of the boundary for \(G_A(t)\).

Let \(A = \{x_1, ..., x_k\}\) and the reliability function of the element \(x_i\) be \(G_i(t)\). Denote by \(A_1, A_2, ..., A_s\) the minimal path sets and by \(C_1, C_2, ..., C_r\) the minimal cut sets
of the subsystem $A$. We have

$$
\prod_{i} [1 - \prod_{j \in C_i} (1 - G_j(t))] \leq G_A(t) \leq 1 - \prod_{i} (1 - \prod_{j \in A_i} G_j(t)).
$$

It is to be expected that the upper (lower) bound should be close to the actual $G_A(t)$ if there is no too much overlap in the minimal path (cut) sets.

In several cases we may compute the probability that the system will work immediately. For example, if the system has a clutter representation and, moreover $\mathcal{P}$ is a family of bases, then it amounts to computing the probability that the related random graphs are connected. Many formulas to calculate such probability are known. If $V$ is a set of vertices of a random graph and $p_{ab}$ is the probability that the edge $\{a, b\}$ exists, i.e. element $\{a, b\} \in X$ of a system works, then denoting by $R(U)$ the probability that the subgraph on the set $U$ of vertices is connected, we have the following formulae.

$$
R(V) = 1 - \sum_{U \subseteq V} R(U) \prod_{a \in U} (1 - p_{ab}),
$$

where $v \in V$ is fixed (for a review see [3]),

$$
R(V) = \sum_{W = V \setminus U \ni w} \left( \prod_{b \in W} (1 - p_{ub})^{-1} - 1 \right) \prod_{a \in U} (1 - p_{ab}) R(U) R(W)
$$

where $u \neq v$ are fixed, (see [9]).

Note that the above formulae can be quite ineffective if a random graph does not have very regular structure, because in the right-hand-side we have to compute up to $2^{|V|}$ summands. Under some assumptions, for example, graph planarity, estimations of $R$ may be obtained, (see [5]).

A special interest in random graph theory is the asymptotic behaviour (i.e. when $|V| \to \infty$) of many properties. However, it seems that the usefulness of such considerations for practical applications in reliability theory is limited.

A different approach to determine $G_A(t)$ is by simulation techniques.

7. Simulation.

The simulation techniques allows us to determine the reliability function $G_A(t)$ of the subsystem consisting of elements of $A$. Assume that the distribution life for the elements at time $t$ is simulated. Next, using the algorithm called "depth first search", the existence of a path joining input and output vertices is tested.
The simulated system "works" when there exists a path joining input and output. We get the estimate

\[ \tilde{G}_A(t) = \frac{\text{number of "working" simulated systems}}{\text{number of simulations}}. \]

This method was applied by the authors in a computer program to calculate the reliability of the systems.

8. Example.

As an example the electronic set containing (1) receiver, (2) type deck, (3) record player, (4) amplifier, (5), (6) speaker and (6) headphone is considered. The family of convex sets consists of

\[ B = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{3, 4, 5, 6\}, \{3, 7\}, \{1, 4, 5, 6\}, \{1, 5, 6\}, \ldots \}. \]

Fig. 2.

In fact, each of the subsets describes a subsystem of the system, which has a parallel-series structure. The most complicated subsystem is characterized by the statement: "some function is realized". Its graph representation for is given on Fig.2.


The proposed approach allows us to make an unified look at compound systems. The analysis of such system has to take into account the functions which are realized by the system and importance of them. The structure of the system changes when the system switches between different functions. This fact is taking into consideration in the proposed model. The constructor can choose which function is most important and tries to augment the probability that this function is realized at the charge of another functions. There are many other situation in which the proposed model is useful.
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