Minimax Control of a Stochastic System
with the Loss Function Dependent
on Parameter of Disturbances

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Summary. A discrete time linear stochastic system with random horizon of control is considered. Disturbances in the system have distribution belonging to an exponential family with a parameter \( \lambda \). The loss function depends on state variables, controls and \( \lambda \) and it is given by (3). A horizon of control is a random variable independent of disturbances with given distribution. For a conjugate a priori distribution of the parameter \( \lambda \) the Bayes control is obtained. Next, a problem of determining the minimax control of the system is considered.


Key words: Bayes control, minimax control, exponential families, linear stochastic systems.

1. Introduction

Let us consider a linear stochastic system defined by equation

\[
    x_{n+1} = \alpha_n x_n + u_n + \gamma_n v_n, \quad x_n \in \mathbb{R}, \quad u_n \in \mathbb{R}, \quad x_0 = c, \quad n = 0, 1, \ldots, N - 1
\]

where \( x_n \) is a state of the system, \( u_n \) is a control, \( v_n \) is a disturbances at time \( n \); \( \alpha_n, \gamma_n \) are given real constants, \( \gamma_n \neq 0 \). In the paper it is assumed that \( v_n, n = 0, 1, \ldots \) are independent, identically distributed random variables with distribution belonging to an exponential family, dependent on an unknown parameter \( \lambda \).

A horizon \( N \) of control is a random variable independent of \( v_0, v_1, \ldots \) with known distribution given by

\[
    P\{N = k\} = p_k, \quad k = 0, 1, \ldots, M; \quad \sum_{k=0}^{M} p_k = 1, \quad p_M > 0.
\]

The following notation is used: \( X_n = (x_0, x_1, \ldots, x_n) \) and \( U_n = (u_0, u_1, \ldots, u_n) \). It is assumed that the control \( u_n \) is based on \( X_n \) and \( U_{n-1} \), then before any data are obtained, the control \( u_n \) is a random variable determined by random disturbances \( v_0, v_1, \ldots, v_{n-1} \) (compare with (1)). For convenience we denote \( U_M \) by \( U \) and we call it a control policy.

Let us assume that a cost of control for a given control policy \( U \) (the loss
function) is given by

\[ L(U, X_\lambda) = \sum_{i=0}^{N} [s_i (x_i - a \lambda)^2 + k w_i^2]. \quad (s_i \geq 0, k \geq 0, a \geq 0, S_M > 0). \] (3)

It means that we want to keep the state of the system at the level proportional to the unknown parameter \( \lambda \). In the sequel, the parameter will be proportional to the expected value of the disturbances. So we want to keep the state of the system at the level proportional to the mean of the disturbances.

The risk connected with the control policy \( U \), when the parameter is equal to \( \lambda \) and the initial state is given, have the form:

\[ R(\lambda, U) = E_\pi E_\lambda [L(U, X_\lambda)] = E_\lambda \left\{ \sum_{i=0}^{N} [s_i (x_i - a \lambda)^2 + k w_i^2] \mid x_0 \right\}. \]

For an a priori distribution \( \pi \) of the parameter \( \lambda \) the expected risk associated with \( \pi \) and a control policy \( U \) is equal to

\[ r(\pi, U) = E_\lambda E_\pi \left\{ \sum_{i=0}^{N} [s_i (x_i - a \lambda)^2 + k w_i^2] \mid x_0 \right\}. \]

(\( E_\pi \) denotes the expectation with respect to distribution of \( N \), \( E_\lambda \) denotes the expectation with respect to the distribution of random variables \( v_0, v_1, \ldots \), when \( \lambda \) is the parameter, \( E_\pi \) denotes the expectation with respect to distribution \( \pi \), \( E_\lambda \) denotes the expectation with respect to joint distribution \( v_0, v_1, \ldots \) and \( \pi \).)

Let the initial state \( x_0 \) and the distribution \( \pi \) of the parameter \( \lambda \) be given. A control policy \( U^* \) is called a Bayes policy when

\[ r(\pi, U^*) = \inf_{U \in \mathcal{D}_\pi} r(\pi, U) \]

where \( \mathcal{D}_\pi \) is the class of the control policies \( U \) for which \( r(\pi, U) \) exists.

We denote by \( \Gamma \) a class of distributions \( \pi \) of the parameter \( \lambda \) and by \( \mathcal{D}_\pi \) the class of control policies \( U \) for which \( r(\pi, U) \) exists for each \( \pi \in \Gamma \). A control policy \( U \in \mathcal{D}_\pi \) is called a minimax control policy with respect to a class of distributions \( \Gamma \) if

\[ \sup_{\pi \in \Gamma} r(\pi, U) = \inf_{U \in \mathcal{D}_\pi} \sup_{\pi \in \Gamma} r(\pi, U). \]

Suppose that disturbances \( v_0, v_1, \ldots \) belong to the exponential family with a parameter \( \lambda \). In the section 3 the problem of determining of the Bayes control for conjugate a priori distribution \( \pi \) of the parameter \( \lambda \) is solved. The problem of minimax control is considered in section 4. Examples of the minimax controls are given in section 5. The section 2 contains some remarks about the exponential family.

The problem of determining of a Bayes control of stochastic systems for disturbances belonging to the exponential family for other cost functions have been considered in [5], [7], [8], [9]. For normally distributed disturbances there are many papers concerning problem of filtration and control of the stochastic
2. The exponential family

In the paper it is supposed that \( \nu_n \) belongs to an exponential family [2], [4], [5], [9], i.e., the density function \( p(v, \lambda) \) with respect to some \( \sigma \)-finite measure \( \mu \) on \( R \) has the form:

\[
p(v, \lambda) = s(v) \exp \left[ qA(\lambda) + vB(\lambda) \right]; \quad q \in \mathbb{Q}^n, \quad \lambda \in \Lambda, \quad v \in R.
\]

(4)

\( A(\lambda) \) and \( B(\lambda) \) are strictly monotonic, analytic function defined on the parameter space \( \Lambda \). We suppose that \( \Lambda \) is a set of all points \( \lambda \) for which

\[
\int_R s(v) \exp \left[ vB(\lambda) \right] d\mu < \infty.
\]

The parametrization function \( A(\lambda) \) and \( B(\lambda) \) are chosen in such a way that

\[
E_\nu \nu_n = q \lambda
\]

and it is assumed that

\[
E_\nu \nu_n^2 = q_1 \lambda^2 + q_2 \lambda + q_3,
\]

(6)

where \( q, q_1, q_2, q_3 \) are constants.

Let \( \nu_n \) has the density given by (4) when \( \lambda \) is the only unknown parameter. Let us suppose the a priori distribution \( \pi \) of \( \lambda \) is conjugate to that one given by (4) i.e. its density has the form

\[
g(\lambda; \beta, r) = c(\beta, r) B(\lambda) \exp \left[ \beta A(\lambda) + rB(\lambda) \right]; \quad \beta \in \mathbb{R}^n, \quad r \in R^n(\beta)
\]

(7)

where \( \beta \) and \( r \) are given parameters such that \( g(\lambda; \beta, r) \) is a density function. For determining the Bayes control the a posteriori density for \( \lambda \) after any new observation must be obtained. As it is known, the a posteriori density of the parameter \( \lambda \) after observing \( X_1 \) and choosing \( \nu_n \) has the same form as (7) i.e. by the Bayes formula

\[
f(\lambda \mid X_1, U_n) = \frac{f(\lambda \mid v_n = v) \cdot p(v, \lambda) g(\lambda; \beta, r)}{\int_{\mathbb{Q}^n} f(\lambda \mid v_n = v) \cdot p(v, \lambda) g(\lambda; \beta, r) d\lambda}
\]

\[
= \frac{B(\lambda) \exp \left[ \beta A(\lambda) + (r + v) B(\lambda) \right]}{\int_{\mathbb{R}^n} B(\lambda) \exp \left[ \beta A(\lambda) + (r + v) B(\lambda) \right] d\lambda}
\]

\[
= C(\beta + q, r + v) B(\lambda) \exp \left[ (\beta + q) A(\lambda) + (r + v) B(\lambda) \right]
\]

\[
g(\lambda; \beta_1, r_1),
\]

where

\[
\beta_1 = \beta + q; \quad r_1 = r + v_n.
\]

Similarly, after \( X_n \) is measured and \( U_{n-1} \) is chosen, the values of \( v_0, v_1, \ldots, v_{n-1} \)
<table>
<thead>
<tr>
<th>Distribution</th>
<th>$\gamma^*$</th>
<th>$Q^*$</th>
<th>$A$</th>
<th>$R^*$</th>
<th>$R^*(\beta)$</th>
<th>$A(\lambda)$</th>
<th>$B(\lambda)$</th>
<th>$q_1$</th>
<th>$q_2$</th>
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<th>$T_1^n$</th>
<th>$T_2^n$</th>
<th>$Q^*$</th>
<th>$Q_1^n$</th>
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<tbody>
<tr>
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<td>{0, 1, \ldots, q}</td>
<td>$N$</td>
<td>(0, 1)</td>
<td>R$^+$</td>
<td>(0, $\beta$)</td>
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<td>$\frac{\lambda}{1-\lambda}$</td>
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<td>$q(1-q)$</td>
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<td>$\frac{1}{\beta_n + 1}$</td>
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<td>$\frac{1}{\beta_n - 1}$</td>
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<td>Negative</td>
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<td>N</td>
<td>R$^+$</td>
<td>(1, -)</td>
<td>R$^+$</td>
<td>$-\ln(1+\lambda)$</td>
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<td>R$^+$</td>
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<td>$\ln \lambda$</td>
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<td>$q^2$</td>
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<td>$\frac{1}{\beta_n}$</td>
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<td>$q$</td>
<td>$q^2$</td>
<td>$q$</td>
</tr>
</tbody>
</table>
are known by (1). It permits to calculate the a posteriori density of $\ell$. It has the form:

$$f(\ell | X_n, U_{n-1}) = \int p(\ell | v_0, v_1, ..., v_{n-1})$$

$$= \int \frac{\beta_n}{\beta_n - v_n} \frac{g(\ell; \beta_n - v_n)}{g(\ell; \beta_n)} d\ell$$

(8)

where

$$\beta_n = \beta + v_n, \quad r_n = r + \sum_{i=0}^{n-1} v_i.$$  

(9)

Let us notice that the distribution (8) does not depend on $U_{n-1}$.

Assume

$$E(\ell | X_n, U_{n-1}) = T^o r_n$$

(10)

and

$$E(\ell^2 | X_n, U_{n-1}) = T^o r_n^2 + T^o r_n + T^o$$

(11)

where $T^o, T^o_i, i = 1, 2, 3$ are constants dependent on $\beta$.

When $X_n$ is given and $U_{n-1}$ is chosen, the conditional distribution of the random variable $v_n$ has the density

$$h(v | X_n, U_{n-1}) = \int A \frac{p(v, \ell) g(\ell; \beta_n, r_n)}{g(\ell; \beta_n, r_n + v)} \frac{d\ell}{C(\ell; \beta_n, r_n)}$$

Suppose that the moments of the random variables $v_n$ for given $X_n, U_{n-1}$ have the form

$$E(v_n | X_n, U_{n-1}) = Q^o r_n$$

$$E(v_n^2 | X_n, U_{n-1}) = Q^o r_n^2 + Q^o r_n + Q^o$$

(12)

for some constants $Q^o, Q^o_i, i = 1, 2, 3$ dependent on the parameter $\beta$.

Examples of distributions belonging to the exponential family are binomial, gamma, negative binomial, normal and Poisson. In Table 1 the values of $q_i, q^2_i, i = 1, 2, 3; T^o, T^o_i, i = 1, 2, 3; T^o, T^o_i, i = 1, 2, 3$ for these distributions are given.

3. The Bayes control

Suppose the disturbances have the distributions with a density given by (4), the a priori distribution of the parameter $\ell$ given by (7) and the initial state is known. Let the distribution of random horizon $N$ be given by (2). Consider the problem of Bayes control for the system described by (1) with starting point at the moment
n, when \( X_n, U_{n-1} \) are given. The expected risk is then:

\[
r_a(\pi, U^n) = \mathbb{E}_X \left\{ \mathbb{E} \left[ \sum_{i=1}^{N} \left[ s_i (x_i - a \lambda)^2 + k_i u_i^2 \right] | X_n, U_{n-1} \right] \bigg| N \geq n \right\}
\]

where \( U^n = (u_n, u_{n+1}, ..., u_M) \).

Let us denote

\[
2 \pi_k = \sum_{i=k}^{M} p_i .
\]

We have

\[
2 r_a = \mathbb{E}_X \left\{ \mathbb{E} \left[ \sum_{i=n}^{N} \left[ s_i (x_i - a \lambda)^2 + k_i u_i^2 \right] | X_n, U_{n-1} \right] \bigg| N \geq n \right\} = \sum_{k=n}^{M} \mathbb{E} \left[ \sum_{i=n}^{k} \left[ s_i (x_i - a \lambda)^2 + k_i u_i^2 \right] | X_n, U_{n-1} \right] \frac{p_k}{\pi_n} 
\]

The Bayes risk for above truncated problem be denoted as

\[
W_a = \inf_{U^n} r_a(\pi, U^n) .
\]

If there exists \( U^{n^*} = (u_n^*, u_{n+1}^*, ..., u_M^*) \) such that

\[
r_a(\pi, U^{n^*}) = W_a
\]

then it is called the Bayes policy for the truncated problem and \( u_i^*, i = n, n+1, ..., M \) are called the Bayes control.

Obviously

\[
r_a(\pi, U^n) = r(\pi, U) \quad \text{and} \quad W_a = r(\pi, U^n) .
\]

For the solution of the Bayes control problem we derive the Bayes controls \( u_i^* \) for \( n = N, N-1, ..., 1, 0 \) recursively. Then \( U^{n^*} \) is the solution of the problem.

When \( X_n, U_{n-1} \) are given the conditional expectation of \( x_{n+1}, r_{n+1}, x_{n+1}', r_{n+1}' \), \( x_{n+1}, r_{n+1} \) can be calculated as follows using (1) and (12).

\[
\mathbb{E}(x_{n+1} | X_n, U_{n-1}) = \mathbb{E} \left[ (x_n x_n + u_n + \gamma_n u_n) | X_n, U_{n-1} \right] 
= x_n x_n + u_n + \gamma_n Q^n r_n 
\]

\[
\mathbb{E}(r_{n+1} | X_n, U_{n-1}) = \mathbb{E} \left[ (r_n + r_n) | X_n, U_{n-1} \right] = r_n (1 + Q^n) 
= \left( \gamma_n (Q^n + Q^n) r_n \right) \frac{1}{2} [ \gamma_n (Q^n + Q^n) r_n + Q^n r_n ] 
\]

\[
\mathbb{E}(x_{n+1}' | X_n, U_{n-1}) = \left( 1 + 2 Q^n + Q^n r_n + Q^n r_n + Q^n r_n + Q^n r_n + Q^n r_n \right) 
= \mathbb{E}(x_{n+1}' | X_n, U_{n-1}) = \gamma_n (Q^n + Q^n) r_n 
\]

\[
\mathbb{E}(x_{n+1}', r_{n+1}' | X_n, U_{n-1}) = \gamma_n (Q^n + Q^n) r_n \frac{1}{2} [ \gamma_n (Q^n + Q^n) r_n + Q^n r_n ] 
\]

Using the backward induction we prove the following lemma.
Lemma 1. For the linear system (1) with the disturbances having the distribution belonging to an exponential family with parameter \( \lambda \) for which an a priori distribution \( N \) is given by (7) and for the random, independent of disturbances bounded horizon \( N \) with distribution (2), the Bayes controls \( u^*_n \) and the Bayes risks \( W_n \) have the form

\[
\begin{align*}
  u^*_n &= -\frac{\pi_{n+1}}{\pi_n} \gamma_n A_n x_n - \frac{\pi_{n+1}}{\pi_n} \gamma_n A_n (Q^n + B_{n+1} (1 + Q^n)) \left( \frac{x_n}{\bar{c}} \right) \\
  W_n &= A_n x_n^2 + 2B_n x_n r_n + C_n x_n^2 + D_n x_n + E_n, \quad n = 0, 1, \ldots, M - 1
\end{align*}
\]

(17)

with coefficients \( A_n, B_n, C_n, D_n, E_n \) satisfying the equations

\[
\begin{align*}
  A_n &= -\frac{\pi_{n+1}}{\pi_n} k_n x_n^2 A_{n+1} \\
  B_n &= -a T^{m} s_n + \frac{\pi_{n+1}}{\pi_n} \gamma_n A_n x_n (Q^n + B_{n+1} (1 + Q^n)) \\
  C_n &= a^2 T^{m} s_n + \frac{\pi_{n+1}}{\pi_n} [\gamma_n Q^n A_{n+1} + 2 \gamma_n (Q^n + Q^n) B_{n+1}] \\
  D_n &= a^2 T^{m} s_n + \frac{\pi_{n+1}}{\pi_n} [A_n + \gamma_n Q^n A_{n+1} + 2 B_n \gamma_n Q^n + C_n Q^n + D_{n+1} (1 + Q^n)] \\
  E_n &= a^2 T^{m} s_n + \frac{\pi_{n+1}}{\pi_n} [A_n + \gamma_n Q^n A_{n+1} + 2 B_n \gamma_n Q^n + C_n (Q^n + E_{n+1})]
\end{align*}
\]

(18)

for \( n = M - 1, \ldots, 0, \) with boundary conditions

\[
\begin{align*}
  A_M &= s_M, \quad B_M = -a T^{m} s_M, \quad C_M = a^2 T^{m} s_M \\
  D_M &= a^2 T^{m} s_M, \quad E_M = a^2 T^{m} s_M
\end{align*}
\]

(19)

Proof. Let \( n = M \) and \( X_M, U_{M-1} \) be given. We have then by (15) and (13)

\[
\begin{align*}
  W_M &= \inf_{u_M} \mathbb{E} \left[ s_M (x_M - \alpha \lambda)^2 + k_M u_M^2 \mid X_M, U_{M-1} \right] \\
  &= s_M [x_M^2 - 2a T^{m} x_M r_M + a^2 (T^{m} r_M + T^{m} r_M + T^{m} r_M)]
\end{align*}
\]

(20)

and the Bayes control \( u_M^* = 0 \).

\( W_M \) is a measurable function and there exists control \( u_M^* = 0 \) which realizes \( W_M \) so we have

\[
W_{M-1} = \inf_{u_{M-1}} \mathbb{E} \left[ \sum_{i=M-1}^{M-1} \frac{\pi_i}{\pi_{M-1}} (s_i (x_i - \alpha \lambda)^2 + k_i u_i^2) \mid X_{M-1}, U_{M-2} \right]
\]
\[
= \inf_{\sigma_{M-1}} \left\{ \sigma_{M-1} \left[ \sigma_{M-1}^2 - 2aT_{M-1}^\sigma \sigma_{M-1}^\sigma \right] + a^2 \left( T_{1}^{M-1} r_{M-1}^\gamma + T_{2}^{M-1} r_{M-1}^\gamma + T_{3}^{M-1} \right) + k_{M-1} \right\}
+ \underbrace{\frac{\pi_{M}}{\pi_{M-1}} \mathbb{E} \left[ \mathbb{E} \left[ \mathbb{E} \left[ s_{M} \left( s_{M} - \alpha \right)^2 \right] \right] \right]}
\]
\[
= \inf_{\sigma_{M-1}} \left\{ \sigma_{M-1} \left[ \sigma_{M-1}^2 - 2aT_{M-1}^\sigma \sigma_{M-1}^\sigma \right] + a^2 \left( T_{1}^{M-1} r_{M-1}^\gamma + T_{2}^{M-1} r_{M-1}^\gamma + T_{3}^{M-1} \right) \right\}
+ \underbrace{\frac{\pi_{M}}{\pi_{M-1}} \mathbb{E} \left[ W_{M-1} | X_{M-1}, U_{M-2}, U_{M-2}^\gamma \right] + k_{M-1} \right\}
\]
\]

From (15) and (20) we have
\[
W_{M-1} = \inf_{\sigma_{M-1}} \left\{ \left( k_{M-1} + \frac{\pi_{M}}{\pi_{M-1}} s_{M} \right) \sigma_{M-1}^2 \right\}
+ 2 \left( 2 \mathbb{E} \left[ \mathbb{E} \left[ \mathbb{E} \left[ s_{M} \left( s_{M} - \alpha \right)^2 \right] \right] \right] \right) s_{M-1}^2 \sigma_{M-1}^2
+ \left( \sigma_{M-1}^2 + \frac{\pi_{M}}{\pi_{M-1}} s_{M}^2 \sigma_{M-1}^2 \right) \sigma_{M-1}^2 + 2 \left( -aT_{M-1}^\gamma \sigma_{M-1}^2 \right)
+ \frac{\pi_{M}}{\pi_{M-1}} s_{M} \left( \sigma_{M-1}^2 - aT_{M}^\gamma \sigma_{M-1}^2 \left( 1 + Q_{M-1}^{\gamma} \right) \right) \sigma_{M-1}^2
+ \left( a^2 T_{1}^{M-1} \sigma_{M-1}^2 + \frac{\pi_{M}}{\pi_{M-1}} s_{M} \left( Q_{M-1}^{\gamma} \right) \right) \sigma_{M-1}^2
- 2aT_{M}^\gamma \sigma_{M-1}^2 \left( Q_{M-1}^{\gamma} \right) + a^2 T_{1}^{M-1} \left( 1 + 2Q_{M-1}^{\gamma} + Q_{M-1}^{2\gamma} \right) \sigma_{M-1}^2
+ \left( a^2 T_{2}^{M-1} \sigma_{M-1}^2 + \frac{\pi_{M}}{\pi_{M-1}} s_{M} \left( Q_{M-1}^{\gamma} - Q_{M-1}^{2\gamma} \right) \right) \sigma_{M-1}^2
+ a^2 T_{3}^{M-1} \sigma_{M-1}^2 + a^2 T_{3}^{M-1} \sigma_{M-1}^2
+ \frac{\pi_{M}}{\pi_{M-1}} s_{M} \left( Q_{M-1}^{\gamma} \right) \sigma_{M-1}^2
+ \left( a^2 T_{1}^{M-1} \sigma_{M-1}^2 + a^2 T_{1}^{M-1} \sigma_{M-1}^2 \right) \sigma_{M-1}^2
\]

Therefore the Bayes control \( u_{M-1}^{\gamma} \) is given by
\[
u_{M-1}^{\gamma} = 2aT_{M}^\gamma r_{M-1}^\gamma + a^2 T_{1}^{M-1} r_{M-1}^\gamma + a^2 T_{1}^{M-1} s_{M-1}^2
\]
and the Bayes risk \( W_{M-1} \) has the form
\[
W_{M-1} = A_{M-1} r_{M-1}^2 + 2B_{M-1} r_{M-1}^\gamma + C_{M-1} + D_{M-1} + E_{M-1}
\]
where the dependencies between $A_{M-1}, B_{M-1}, C_{M-1}, D_{M-1}, E_{M-1}$ and $A_M, B_M, C_M, D_M, E_M$ are easy to establish to obtain (18) for $n = M - 1$.

Let us assume for induction that the truncated Bayes risk $W_i, i = n + 1, n + 2, \ldots, M$ have the form (17), the Bayes controls $u_i^\pi, i = n + 1, \ldots, M$ exist and they have the form (17), where $A_i, B_i, C_i, D_i, E_i$ satisfy (18). We have then

\[
W_n = \inf_{u_n} \{E [s_n (x_n - a \hat{\lambda})^2 + k_n u_n^2 \mid X_n, U_{n-1}] + \inf_{\theta \neq \bar{\pi}} \frac{\theta}{\bar{\pi}} E \left[ \sum_{i=n+1}^{M} \frac{\theta}{\bar{\pi}} [s_i (x_i - a \hat{\lambda})^2 + k_i u_i^2] X_{n+1}, U_n \right] \mid X_n, U_{n-1} \},
\]

and from induction assumptions we have

\[
W_n = \inf_{u_n} \left\{ s_n [x_n^2 - 2aT^m x_n + a^2 (T^m r_n + T^m T^m r_n + T^m)] + k_n u_n^2 \right\} + \frac{\pi_n+1}{\bar{\pi}} E \left[ W_{n+1} \mid X_n, U_{n-1} \right].
\]

From (17) and (15) we calculate $E \left[ W_{n+1} \mid X_n, U_{n-1} \right]$ and using (21) we determine $W_n$, the Bayes control $u_n^\pi$ and the relations between $A_n, B_n, C_n, D_n, E_n$ and $A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1}, E_{n+1}$. They satisfy the equations (17), (16), and (18) respectively.

4. The minimax control

Now, we are in the position to look for the minimax control $\hat{U}$ for $\mathcal{U}$ being the class of distribution $\pi$ such that $E_\pi x^2 = m_2$, where $m_2$ is known. At the first let us consider the situation when the parameter $\lambda$ and the control policy is given. Then the distribution of disturbances $\mathcal{e}_n, \mathcal{r}_n, \ldots$ are given and the risk $R(\hat{\lambda}, U)$ can be calculated. The truncated risk $R_n(\hat{\lambda}, U)$ we define as

\[
R_n(\hat{\lambda}, U) = E_X \left\{ \sum_{i=n}^{N} \left\{ s_i (x_i - a \hat{\lambda})^2 + k_i u_i^2 \right\} \mid X_n, U_{n-1} \right\} \mid X \equiv \mathcal{N} \right\}.
\]

This may be transformed to

\[
R_n(\hat{\lambda}, U) = E_X \left\{ \sum_{i=n}^{N} \left\{ s_i (x_i - a \hat{\lambda})^2 + k_i u_i^2 \right\} \mid X_n, U_{n-1} \right\} \mid X \equiv \mathcal{N} \right\}.
\]

For the Bayes controls $u_i^\pi, i = 0, 1, 2, \ldots$ with respect to the distribution $\pi$ of the parameter $\lambda$ given by (7) the risk will be denoted by $R_n$. $R_n$ fulfills the following recurrence relation:

\[
R_n = s_n (x_n - a \hat{\lambda})^2 + k_n u_n^2 + \frac{\pi_n+1}{\bar{\pi}} E_X [R_{n+1} \mid X_n, U_{n-1}]
\]

for $n = M - 1, M - 2, \ldots, 0$.

\[
R_M = s_M (x_M - a \hat{\lambda})^2 + k_M u_M^2.
\]
This equation can be proved as follows:

\[ R_n = E_1 \left[ \sum_{i=n}^{M} \frac{\pi_i}{\pi_n} \left( s_i(x_i - \alpha \lambda)^2 + kr^2 \right) | X_n, U_{n-1} \right] \]

\[ = E_1 \left[ s_n(x_n - \alpha \lambda)^2 + kr^2 | X_n, U_{n-1} \right] \]

\[ + \frac{\pi_{n+1}}{\pi_n} E_1 \left[ \sum_{i=n+1}^{M} \frac{\pi_i}{\pi_{n+1}} \left( s_i(x_i - \alpha \lambda)^2 + kr^2 \right) | X_{n+1}, U_n \right] | X_n, U_{n-1} \right] \]

\[ = s_n(x_n - \alpha \lambda)^2 + kr^2 + \frac{\pi_{n+1}}{\pi_n} E_1 \left[ R_{n+1} | X_n, U_{n-1} \right]. \]

For the Bayes control \( u^*_n \), \( n = 0, 1, ..., M \) the truncated risk have the form given by lemma 2.

**Lemma 2.** If the disturbances \( e_0, e_1, \ldots \) have the distribution given by (4) and the Bayes control \( u^*_n \), \( n = 0, 1, ..., M \) with respect to the distribution (7) of the parameter \( \lambda \) is given by (16), then the truncated risk \( R_n \) for the Bayes policy \( U^* \) has the form:

\[ R_n = a_n e_n^2 + c_n r_n^2 + 2d_n x_n \lambda + e_n \lambda^2 + 2f_n r_n \lambda + i_n \lambda + j_n, \]

where \( a_n, c_n, d_n, e_n, f_n, i_n, j_n \) satisfy the equations

\[ a_n = A_n, \]

\[ c_n = k_n \eta_n + \frac{\pi_{n+1}}{\pi_n} (a_n + r_n k + c_{n+1}), \]

\[ d_n = -as_n + \frac{\pi_{n+1}}{\pi_n} (a_n + r_n k + d_{n+1} \xi_n), \]

\[ e_n = a^2 \beta_n + \frac{\pi_{n+1}}{\pi_n} (a_n + r_n k + e_{n+1} + 2d_n + r_n k + e_n + 2f_n + i_n), \]

\[ f_n = \frac{\pi_{n+1}}{\pi_n} (a_n + r_n k + f_{n+1} + i_n), \]

\[ i_n = \frac{\pi_{n+1}}{\pi_n} (a_n + r_n k + i_{n+1}), \]

\[ j_n = \frac{\pi_{n+1}}{\pi_n} (a_n + r_n k + j_{n+1}). \]

\( \xi_n \) and \( \eta_n \) denote

\[ \xi_n = \frac{\beta_n k}{k + \frac{\pi_{n+1}}{\pi_n} A_{n+1}} \]

\[ \eta_n = \frac{\pi_{n+1} Q_{n+1} + P_{n+1} (1 + Q_n)}{k + \frac{\pi_{n+1}}{\pi_n} A_{n+1}} \]

with conditions

\[ a_M = s_M, \quad d_M = -s_M \theta, \quad e_M = s_M \theta^2, \quad f_M = i_M = j_M = 0. \]

**Proof.** Let us assume that \( R_n \) has the form:

\[ R_n = a_n e^2 + 2d_n x_n \lambda + e_n \lambda^2 + 2f_n r_n \lambda \]

\[ + g_n x_n + h_n r_n + i_n \lambda + j_n, \]

\[ \lambda \text{ or } \lambda^2 \text{ or } \lambda^3. \]
From (8), (5) and (6) for a control \( u_n \) we have
\[
E_I(x_{n+1} | X_n, U_{n-1}) = E_I \left( \left( \alpha_n x_n + u_n + \gamma_n u_n \right) | X_n, U_{n-1} \right) \\
= \alpha_n x_n + u_n + \gamma_n u_n g \lambda
\]
\[
E_I(r_{n+1} | X_n, U_{n-1}) = E_I \left( r_n + v_n | X_n, U_{n-1} \right) = r_n + q \lambda
\]
\[
E_I(x_{n+1} | X_n, U_{n-1}) = (\alpha_n x_n + u_n)^2 + 2 (\alpha_n x_n + u_n) \gamma_n q \lambda \\
+ \gamma_n^2 (q \lambda^2 + q^2 \lambda + q \beta)
\]
\[
E_I(x_{n+1} | X_n, U_{n-1}) = (\alpha_n x_n + u_n) r_n + (\alpha_n x_n + u_n) q \lambda \\
+ \gamma_n x_n q \lambda + \gamma_n (q \lambda^2 + q^2 \lambda + q \beta)
\]
\[
E_I(x_{n+1} | X_n, U_{n-1}) = r_n^2 + 2q r_n \lambda + q \lambda^2 + q^2 \lambda + q \beta
\]

For the Bayes control \( u_n^+ \) using (24) and (25) we obtain:
\[
E_I(R_{n+1} | X_n, U_{n-1}) = a_{n+1} \left( (\alpha_n x_n + u_n)^2 + 2 (\alpha_n x_n + u_n) \gamma_n q \lambda \\
+ \gamma_n^2 (q \lambda^2 + q^2 \lambda + q \beta) \right) \\
+ 2\gamma_n \left( (\alpha_n x_n + u_n) (r_n + q \lambda) + \gamma_n r_n q \lambda + \gamma_n (q \lambda^2 + q^2 \lambda + q \beta) \right) \\
+ c_{n+1} (r_n^2 + 2q r_n \lambda + q \lambda^2 + q^2 \lambda + q \beta) \\
+ 2 \gamma_n (\alpha_n x_n + u_n + \gamma_n q \lambda) + e_{n+1} \lambda^2 + 2f_{n+1} (r_n + q \lambda) \lambda \\
+ g_{n+1} (\alpha_n x_n + u_n + \gamma_n q \lambda) + h_{n+1} (r_n + q \lambda) \\
+ i_{n+1} \lambda + j_{n+1}
\]

For \( n = M \) the relations (22) and (16) give
\[
R_M = \alpha_M (x_M^2 - 2r_M a \lambda + a^2 \lambda^2)
\]
This yields
\[
a_M = \alpha_M, \quad d_M = -\alpha_M a, \quad e_M = \alpha_M a^2,
\]
\[
b_M = c_M = f_M = g_M = h_M = i_M = j_M = 0.
\]

Let us denote
\[
\alpha_n x_n + u_n^2 = \tilde{z}_n x_n + \eta_n u_n
\]
(28)

where
\[
\tilde{z}_n = \frac{\alpha_n k_n}{k_n + \frac{r_{n+1}}{\alpha_n} A_{n+1}} \quad \eta_n = \frac{\frac{r_{n+1}}{\alpha_n} A_{n+1} (Q^{(n)} + R_{n+1} (1 + Q^{(n)}))}{k_n + \frac{r_{n+1}}{\alpha_n} A_{n+1}}
\]
for \( n = 0, 1, 2, ..., M - 1 \).

Substituting \( E_I[R_{n+1} | X_n, U_{n-1}] \) by (26) in (22), taking into account (25) and comparising the terms containing \( x_n^2, x_n r_n, x_n \lambda, \lambda^2, r_n \lambda, x_n, r_n \), 1 we obtain the equations
\[
a_n = a_n + \tilde{z}_n \frac{r_{n+1}}{\alpha_n} \eta_n + k_n (\tilde{z}_n - z_n)^2
\]
\[
b_n = \frac{r_{n+1}}{\alpha_n} (a_n + \tilde{z}_n \eta_n) + b_n + \tilde{z}_n \frac{1}{\alpha_n} + k_n (\tilde{z}_n - z_n) \eta_n
\]
\begin{align*}
c_n &= \frac{\pi_{n+1}}{\pi_n} \left[ a_n + (\gamma_n + 2b_n + i_n + c_n + 1) \right] + k_n q_n^2 \\
d_n &= \frac{\pi_{n+1}}{\pi_n} \left[ a_n + (\gamma_n + 2b_n + i_n + c_n + 1) \right] - s_n t_n \\
e_n &= \frac{\pi_{n+1}}{\pi_n} \left[ a_n + (\gamma_n + 2b_n + i_n + c_n + 1) \right] + 2f_n + y_n + s_n q_n^2 \\
f_n &= \frac{\pi_{n+1}}{\pi_n} \left[ a_n + (\gamma_n + 2b_n + i_n + c_n + 1) \right] + 2f_n + y_n + s_n q_n^2 \\
g_n &= \frac{\pi_{n+1}}{\pi_n} \left[ a_n + (\gamma_n + 2b_n + i_n + c_n + 1) \right] \\
h_n &= \frac{\pi_{n+1}}{\pi_n} \left[ a_n + (\gamma_n + 2b_n + i_n + c_n + 1) \right] \\
i_n &= \frac{\pi_{n+1}}{\pi_n} \left[ a_n + (\gamma_n + 2b_n + i_n + c_n + 1) \right] + 2f_n + y_n + s_n q_n^2 \\
j_n &= \frac{\pi_{n+1}}{\pi_n} \left[ a_n + (\gamma_n + 2b_n + i_n + c_n + 1) \right] + 2f_n + y_n + s_n q_n^2 \\
\end{align*}

From the above equations and (18) we have

\begin{align*}
a_n - A_n &= \frac{\gamma_n^2}{\pi_n} \left[ a_n + (\gamma_n + 2b_n + i_n + c_n + 1) \right]
\end{align*}

but \(A_0 = \theta_0 - s_0\), then \(a_n = A_n\) for \(n = 0, 1, \ldots, M\).

The second row of (29) yields \(b_n = 0\), \(n = 0, 1, \ldots, M\). The conditions (27) and equations (29) give also \(g_n = h_n = 0\) for \(n = 0, 1, \ldots, M\). So the lemma 2 is proved.

We call a control policy \(U\) to be a constant risk control policy if \(R(\lambda, U) = \text{const}\) for each \(\lambda \in \Lambda\).

**Lemma 3.** [10] The constant risk policy \(U\) which is the Bayes one with respect to some \(\pi \in \Gamma\) is the minimax control policy.

Since the risk for the Bayes control policy \(U^*, \ R_0 = R(\lambda, U^*)\), has the form given by lemma 2 and we assume that \(E_0 \lambda^2 = m_2\), for each \(\pi \in \Gamma\), then \(U^*\) will be the constant risk policy if

\begin{align*}
2d_0 x_0 + 2f_0 x_0 + t_0 &= 0.
\end{align*}

Let \(\pi^*\) be the distribution of the parameter \(\lambda\) which has the density (7) and for which the control policy \(U^*\) is the Bayes one. Since \(\pi^* \in \Gamma\) and (11) holds we obtain that the parameters \(\alpha^*\) and \(\beta^*\) of distribution \(\pi^*\) satisfy the condition

\begin{align*}
T_{\alpha^*} + T_{\beta^*} + T_{\beta^*} &= m_2.
\end{align*}

where \(T_i = T_i(\beta^*), \ i = 1, 2, 3\).

Lemma 3 and the above remarks lead to the following theorem.

**Theorem 1.** Let for given \(x_0 = c\), the values \(\beta^*, \alpha^*\) satisfy the equations (30) and (31), where \(d_0\) is a constant, \(f_0 = f_0(\beta^*), \ t_0 = t_0(\beta^*)\) are the function of \(\beta^*\) determined from the
recursive formulae (18) and (24); \( Q^p = Q^p(\beta) \) is given by (12). The policy \( U^* \), the Bayes one with respect to the distribution \( \pi^* \) defined by (7) with \( \beta = \beta^* \), \( r = r^* \), for \( x_0 = c \), is a minimax control policy. This control policy is given by (16) with \( r_n = r^* + \sum_{i=0}^{n-1} \nu_i, Q^p = Q^p(\beta^*) \).

When the class \( \Pi \) contains all the a priori distributions of the parameter \( \lambda \) or when we do not determine \( \Pi \) at all, we define the minimax control policy \( \hat{U} \) as follows [10]. \( \hat{U} \) is the minimax control policy if

\[
\sup_{\lambda \in \Pi} R(\lambda, \hat{U}) = \inf_{U \in \mathcal{U}} \sup_{\lambda \in \Pi} R(\lambda, U),
\]

where \( \mathcal{U} \) is a class of controls for which \( R(\lambda, U) \) is finite for all \( \lambda \).

The lemmas 2 and 3 permit to obtain the minimax controls also in this situation.

**Theorem 2.** Let for given \( x_0 = c \), the values \( \beta^* \), \( r^* \) satisfy the equations

\[
\begin{align*}
e_0 &= 0 \\
2d_0r_0s_0 + 2f_0q_0 + i_0 &= 0 \\
(32)
\end{align*}
\]

where \( d_0, e_0 \) are constants; \( f_0 = f_0(\beta), i_0 = i_0(\beta) \) are the functions of \( \beta \) determined recursively from (18) and (24); \( Q^p = Q^p(\beta) \) is given by (12). The policy \( U^* \), the Bayes one with respect to the distribution (7), with \( \beta = \beta^*, r = r^* \), for \( x_0 = c \) is a minimax control policy. This control policy is given by (16) with \( r_n = r^* + \sum_{i=0}^{n-1} \nu_i, Q^p = Q^p(\beta^*) \).

5. The minimax controls for \( N = 1 \)

The equations (18), (24) permit to obtain, when \( N = 1 \) with probability 1, the required constants \( d_0, e_0, f_0, i_0 \)

\[
\begin{align*}
d_0 &= \frac{\sigma_0 \psi_0 - a}{k_0 + s_1} (\gamma_0 q_0 - a) - s_0 a = a^2 s_0 + s_1 (\gamma_0 q_0 - 2a^2 q_0 + a^2) \\
f_0 &= -s_1^2 (\gamma_0 q_0 - a) \frac{\sigma_0 q_0 - T_0 q_0 (1 + q_0^2)}{k_0 + s_1} = s_1^2 q_0 q_2 .
\end{align*}
\]

Let us suppose (see table 1) that \( Q_0 = \frac{q}{\beta} \) and \( T_0 = \frac{1}{\beta + q} \) then

\[
f_0 = -s_1^2 \frac{(\gamma_0 q_0 - a)^2}{(k_0 + s_1) \beta}
\]

and the equation (30) takes the form:

\[
2 \left[ \frac{\sigma_0 \psi_0 (\gamma_0 q_0 - a)}{k_0 + s_1} - s_0 q_0 \right] x_0 = \frac{2s_1^2 (\gamma_0 q_0 - a)^2 r^*}{k_0 + s_1} \frac{r^*}{\beta^* - s_1^2 q_0^2} . \tag{33}
\]

Let us consider the minimax control in the situation when \( E_{\lambda} \lambda^2 = m_2, m_2 \) is given. From the theorem 1 the minimax control for this class exists when besides
(33) also the equation
\[ T_{0}^{s_{2}} + T_{0}^{s} + T_{1}^{s} = m_{2} \]
holds.

Let us take the equation (33) in consideration.

A. The analysis of this equation gives that there are not restriction for the initial state \( x_{0} \) when \( a = 0 \) and, \( \gamma_{0} = 0 \) or \( q_{2} = 0 \) (i.e. for normal and gamma distribution) or when \( \gamma_{0} a \neq 0 \), \( x_{0} k_{0} s \) \((\gamma_{0} a - a) = s_{0} a (k_{0} + s_{1})\) and \( \frac{r^{s}}{\beta^{s}} = \frac{\gamma_{0} a (k_{0} + s_{1})}{s_{1} (\gamma_{0} a - a)^{2}} \) for \( r^{s} \in R^{s}, \beta^{s} \in B^{s} \).

B. When \( \gamma_{0} a = a, a \neq 0 \) the initial state \( x_{0} \) for which the minimax control can be obtained is independent of the parameters \( r^{s}, \beta^{s} \) which must satisfy the equation (31). There are in general many solutions \( (r^{s}, \beta^{s}) \) of the equation (31). Each control \( u^{s} \) which is obtained by using these pairs is minimax one if only \( x_{0} \) fulfills the equation
\[ x_{0} = \frac{s_{0} r_{0}^{s}}{2s_{0} a}. \]

C. When \( q + a \) and \( s_{0} k_{0} s_{1} (\gamma_{0} a - a) = s_{0} a (k_{0} + s_{1}) \) the initial state \( x_{0} \) for which the minimax controls can be obtained is determined by the solution of the equation (31) and
\[ x_{0} = \frac{s_{1} r_{0}^{s} (\gamma_{0} a - a)^{2}}{s_{0} k_{0} s_{1} (\gamma_{0} a - a) - s_{0} a (k_{0} + s_{1}) \beta^{s}} = \frac{s_{1} r_{0}^{s} (k_{0} + s_{1})}{s_{0} k_{0} s_{1} (\gamma_{0} a - a) - s_{0} a (k_{0} + s_{1})} \cdot \]

For the given disturbances we can investigate the equation (31) more precisely, for example the proportion \( \frac{r^{s}}{\beta^{s}} \) takes the values in the intervals \( (m_{2}, \sqrt{m_{2}}) \) for the binomial distribution; in \( (0, \sqrt{m_{2}}) \) for the gamma, negative binomial and Poisson one, and in \( (-\sqrt{m_{2}}, \sqrt{m_{2}}) \) for the normal distribution.

Then the minimax control exists (and may be given explicit) if the ratio \( \frac{r^{s}}{\beta^{s}} \) from the equation (33) belongs to the corresponding interval.

Now, let us consider the problem of determining of the minimax policy when all a priori distributions are admitted or if there are no restriction of them. Among the distributions considered in table 1 only in case the binomial case the solution can exist. For this distribution the first equation of (32) takes the form:
\[ a s_{0} + s_{1} (\gamma_{0} a q - 2a \gamma_{0} a + a^{2}) = 0. \]  

(34)

If for fixed other constants there exists a natural number \( q \) satisfying (34) and the ratio \( \frac{r^{s}}{\beta^{s}} = (0, \beta^{s}), \beta^{s} > 0 \) then the solution of (33) with \( q_{2} = q \), then for the binomial distribution with this parameter \( q \) each control of the form (16) determined by \( r^{s}, \beta^{s} \) is a minimax control.

Let for example \( s_{0} = s_{1} = \gamma_{0} = k_{0} = a = 1 \). Then for \( a = 1 \) we have two solutions \( q = 2 \) and \( q = 1 \). In the first case for each \( x_{0} \in (1, 2) \) there exists minimax control.

In the second one we have the minimax control if \( x_{0} = \frac{1}{2} \) only.
If $a = 0$, i.e. the loss function does not depend on the parameter $z$ we have one solution $q = 1$ (i.e. for two points distribution). In this case the minimax control exists if $x_0 \in (-2, 0)$.

When $a = 0$, $\gamma_0 = 2$, $\psi_0 = s_1 = 1$, $s_0 = k_0 = 0$ we are in the situation A. The minimax control for each initial state $x_0$ is

$$u_0 = -\left(2x_0 + \frac{1}{2}\right),$$

the same as it is given in Aoki [1, Example 8].

References


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Das vorliegende Buch versucht, die bekanntesten mathematischen Methoden zur Behandlung von Problemen der Systemzuverlässigkeit auf einem für Praktiker zugänglichen mathematischen Niveau darzustellen und anhand von vielen Beispielen und heuristischen Erläuterungen plausibel zu machen. Diesem Anspruch wird das Buch sehr gut gerecht. Auf mathematische Ableitungen wird größtenteils verzichtet; auch liegt das Gewicht nicht bei einer Aufstellung sofort anwendbarer Formeln für spezielle Systeme. Im Mittelpunkt steht vielmehr das Heransuchen an mathematische Modelle und Lösungsmethoden.

Da die notwendigen Grundlagen aus der Wahrscheinlichkeitsrechnung und spezieller Fragen der Theorie zufälliger Prozesse recht umfassend in den Ergebnissen dargestellt werden und dabei stets auf die Bezüge zu zuverlässigkeits theoretischen Fragestellungen hingewiesen wird, ist das Buch sehr gut dazu geeignet, sich in die mathematische Zuverlässigkeitsrechnung einzuarbeiten. Formal genügen für das Verständnis analytische Kenntnisse bis zur Differential- und Integralrechnung, jedoch sollte ein Grundkurs in Wahrscheinlichkeitsrechnung vorangestellt werden.

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