MINIMAX CONTROL OF A SECOND ORDER LINEAR SYSTEM

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ABSTRACT

A discrete time second order linear stochastic system with a random horizon is considered. Disturbances on the system have a distribution belonging to an exponential family with parameter \( \lambda \). The loss function is quadratic, non-negatively defined, dependent on state variables, controls and \( \lambda \). The horizon of the control is a random variable with a given distribution, independent of disturbances. In this paper a minimax control for such system is obtained.

1. Preliminary

Models of an adaptive stochastic control theory were described by Aoki [1]. These models were generalized to a non-stationary Bayes dynamic decision model by Rieder [6] and to a model of the Bayes control of the Markov chain by van Hee [3]. In the cited papers the existence of the Bayes control was shown. The explicit solution of the problem of the Bayes control of a linear stochastic system with a fixed horizon and the disturbances having Poisson, binomial or negative binomial distributions was given by Trybula [7]. A minimax control for the disturbances belonging to the exponential class with the variance being the quadratic function of the mean was considered also by Trybula [8].

In mechanics, electricity and many other fields, the second order system are of frequent occurrence. Therefore the analyses of these systems in every detail are motivated by practical applications. The Bayes control for the second order linear system (SOLS) with the random horizon has been obtained as an example in [5]. In this paper, after a short introduction, the Bayes control is represented and the problem of the minimax control is solved. In the last section some remarks about the higher order linear systems and the minimax controls for them are given.
2. Introduction

Let us consider an adaptive optimal control problem of the SOLS defined by the equation

\[ x_{n+1} = a_{2,n} x_n + a_{1,n} x_{n-1} + \delta_n u_n + \gamma_n v_n, \quad x_n \in R, \quad u_n \in R, \]
\[ n = 0, 1, \ldots, N-1. \tag{1} \]

where \( x_n \) is a state of the system, \( u_n \) is a control, \( v_n \) is a disturbance at the moment \( n \); \( a_{1,n}, a_{2,n}, \delta_n, \gamma_n \) are given real constants, \( \gamma_n \neq 0 \) and \( R \) denotes the real line. It is assumed that \( v_n, n = 0, 1, \ldots \) are independent and identically distributed random variables with a distribution belonging to an exponential family (see [2]) dependent on an unknown parameter \( \lambda \in \Lambda \) with the finite variance being the quadratic function of the mean. \( \Lambda \) is the natural parameter space. It is known (see [4]) that there is only six distributions (and the linear transformations of them) having such properties, i.e., the binomial, GEHS (generalized exponential hyperbolic secant), gamma, negative binomial, normal and Poisson distribution. The parametrization is chosen in such a way that

\[ E_\lambda v_n = q, \tag{2} \lambda, \]

and

\[ E_\lambda v_n^2 = q (q - \xi) \lambda^2 + \eta q \lambda + v, \tag{3} \]

where \( q \) is a given constant, \( \xi \) is equal to 1 for the binomial distribution, 0 for the normal or Poisson distribution, and \(-1\) otherwise; \( \eta \) is equal to 1 for the binomial or negative binomial or Poisson distribution and 0 otherwise; \( v \) is equal to 1 for the normal or GEHS distribution and 0 otherwise.

A horizon of the control is a random variable independent of \( v_0, v_1, \ldots \), with a known distribution given by

\[ P \{ N=k \} = p_k, \quad k=0, 1, \ldots, M, \quad \sum_{k=0}^{M} p_k = 1, \quad p_M > 0. \tag{4} \]

Denote

\[ p_n = \sum_{i=n}^{M} p_i. \]

The following notations are used: \( X_n = (x_{n-1}, x_0, x_1, \ldots, x_n) \) and \( U_n = (u_0, u_1, \ldots, u_n) \). It is assumed that the control \( u_n \) is based on \( X_n \) and \( U_{n-1} \), then before any data are obtained, the control \( u_n \) is a random variable determined by the random disturbances \( v_0, \ldots, v_{n-1} \) (see [1]). For convenience we denote \( U^n = (u_0, u_{n+1}, \ldots, u_M) \) and \( U' = U_M - U^0 \).
Let us assume that the cost of the control for a given control policy $U$ is given by

$$J(U, X_N) = \sum_{i=0}^{N} (s_i x_i^2 + 2t_i x_i \lambda + w_i x_i^2 + k_i u_i^2),$$  \hspace{1cm} (5)

where $s_i, w_i, t_i, k_i > 0$, $s_i > 0$, $s_M > 0$, $w_i > 0$ for each $i$.

The risk connected with the control policy $U$, when the parameter is $\lambda$ and $x_{-1}, x_0$ are given, is defined as

$$R(\lambda, U) = E_N \{E_{\lambda} [J(U, X_N) | x_{-1}, x_0] \},$$

where $E_N$ denotes the expectation with respect to the distribution (4) of the random variable $N$; $E_{\lambda}$ denotes the expectation with respect to the distribution of the random variables $\nu_u, \nu_u = 0, 1, \ldots, N-1$, for fixed $\lambda$.

Denote $D = \{U: \varphi(\lambda, U) < \infty\}$, A policy $\hat{u}$ such that

$$\sup_{\lambda \in \Lambda} R(\lambda, \hat{u}) = \inf_{U \in D} \sup_{\lambda \in \Lambda} R(\lambda, U),$$

is called the minimax policy.

Previous observations of the disturbances frequently give information about the parameter $\lambda$. We can use them to reduce the costs of the control. Let the information about $\lambda$ be given by prior distributions $\pi$ of the parameter $\lambda$. Denote by $\Gamma$ the set of all such distributions.

The expected risk, where $\pi$ is given and $U$ is chosen, is defined as

$$r(\pi, U) = E_\pi R(\lambda, U),$$

where $E_\pi$ denotes the expectation with respect to the distribution $\pi$.

Let $\Gamma_0 \subset \Gamma$. A policy $U$ such that

$$\sup_{\pi \in \Gamma_0} r(\pi, U) = \inf_{U \in D} \sup_{\pi \in \Gamma_0} r(\pi, U)$$

is called the $\Gamma_0$-minimax policy. When $\Gamma_0$ contains only one distribution the control policy $U$ is denoted by $U^*$ and it is called the Bayes control policy. The Bayes control policies play an important role in obtaining the minimax policies. The $\Gamma$-minimax policy is the minimax one.

3. A Bayes Control

Denote by $\pi_{\beta, r}$ the conjugate prior distribution with parameters $(\beta, r) \in \mathcal{S}$ to the distribution of the disturbances (see [2], [5], [8]). The
set $S$ of the parameters $(\beta, r)$ is such that $\pi_{\beta, r}$ is well defined. If the distribution $\pi_{\beta, r}$ is assigned to $\lambda$ then the posterior distribution of $\lambda$, given $X_n, U_{n-1}$, is $\pi_{\beta_n, r_n}$, where

$$\beta_n = \beta + nq; \quad r_n = r + \sum_{i=0}^{n-1} \gamma_i$$

(see [5], [8]).

Let us assume that the second moment of $\pi_{\beta, r}$ is finite and denote $S^2 = \{ (\beta, r) \in S: E_{\pi_{\beta, r}} \lambda^2 < \infty \}$. Then we have

$$E_{\pi_{\beta, r}} (\lambda^2 \mid X_n, U_{n-1}) = \frac{r_n}{\beta_n^2} + \frac{\eta}{\beta_n + \xi},$$

$$E_{\pi_{\beta, r}} (\lambda^2 \mid X_n, U_{n-1}) = T_1^n \left( \frac{r_n}{\beta_n} \right)^2 + T_2^n \left( \frac{r_n}{\beta_n} \right) + T_3^n,$$

where

$$T_1^n = \frac{\eta_n}{\beta_n + \xi},$$

$$T_2^n = \frac{\eta}{\beta_n + \xi},$$

$$T_3^n = \frac{\eta}{\beta_n + \xi},$$

and the set $S^2$ is: $\{ \beta > 0, \ r > 0 \}; \{ \beta > 1, \ r > 0 \}; \{ \beta > 1, \ -\infty < r < \infty \}; \{ \beta > 1, \ r > 0 \}; \{ \beta > 0, \ -\infty < r < \infty \}; \{ \beta > 0, \ r > 0 \}$ for the binomial, gamma, GEHS, negative binomial, normal and Poisson distribution, respectively.

The conditional expectations of the disturbances have the form

$$E (v_n \mid X_n, U_{n-1}) = q \frac{r_n}{\beta_n},$$

$$E (v_n^2 \mid X_n, U_{n-1}) = Q_1^n \left( \frac{r_n}{\beta_n} \right)^2 + Q_2^n \left( \frac{r_n}{\beta_n} \right) + Q_3^n,$$

where

$$Q_1^n = q(q-\xi) T_1^n,$$

$$Q_2^n = q(q+\beta_n) T_2^n,$$

$$Q_3^n = q(q+\beta_n) T_3^n$$

(see [5], [8]). $E$ is taken with respect to the joint distribution $v_n, v_1, \ldots$ and $\lambda$.

From [5] the following lemma is satisfied.

**Lemma 1.** For the above described SOLS the Bayes control policy has the form:

$$u_{n^*}^* = 0,$$

$$u_n^* = \epsilon_n X_n + X_n X_{n-1} + r_n \frac{r_{n-1}}{\beta_n}, \ n = 0, 1, \ldots, M-1$$

(8)
where
\[ \epsilon_n = \frac{\rho_n + 1}{\rho_n} \delta_n \frac{A_{n+1} \epsilon_{n+1} + B_{n+1}}{\delta_n A_{n+1}} \]
\[ x_n = \frac{\rho_n + 1}{\rho_n} \delta_n \frac{A_{n+1} \epsilon_{n+1} + B_{n+1}}{\delta_n A_{n+1}} \]
\[ \tau_n = \frac{\rho_n + 1}{\rho_n} \delta_n \frac{A_{n+1} \epsilon_{n+1} + B_{n+1}}{\delta_n A_{n+1}} \]

The constants \( A_n, B_n, D_n \) satisfy the equations:
\[ A_n = s_n + k_n \epsilon_n^2 + \frac{\rho_{n+1}}{\rho_n} [A_{n+1} (a_{2,n} + s_n \epsilon_n) + 2B_{n+1} (a_{2,n} + s_n \epsilon_n) + C_{n+1}] \]
\[ B_n = k_n x_n \epsilon_n + \frac{\rho_{n+1}}{\rho_n} [A_{n+1} (a_{2,n} + s_n \epsilon_n) (a_{1,n} + s_n x_n) + B_{n+1} (a_{1,n} + s_n x_n)] \]
\[ C_n = k_n x_n^2 + \frac{\rho_{n+1}}{\rho_n} A_{n+1} (a_{1,n} + s_n x_n)^2 \]
\[ D_n = t_n + \frac{\rho_{n+1}}{\rho_n} [A_{n+1} (a_{2,n} + s_n \epsilon_n) \gamma_n q + B_{n+1} \gamma_n q + D_{n+1} (a_{2,n} + s_n \epsilon_n) + E_{n+1}] \]
\[ E_n = \frac{\rho_{n+1}}{\rho_n} [A_{n+1} \gamma_n q (a_{2,n} + s_n x_n) + D_{n+1} (a_{1,n} + s_n x_n)] \]

with the boundary conditions
\[ A_M = s_M, D_M = t_M, B_M = C_M = E_M = 0. \]

For further considerations we calculate the risk function for the Bayes control \( U^*_n, \beta \neq nq; n = 0, 1, \ldots, M-1 \). In order to calculate the risk function the truncated problem of the control is considered, i.e., the problem of the control for the system described by (I) with the starting point at the moment \( n \), when \( X_n \) and \( U_{n-1} \) are given. The risk is then:
\[ R_n (\lambda, U^n) = E_N \{ \sum_{i=1}^{N} (s_i x_i^2 + 2t_i x_i \lambda + w_i \lambda^2 + k_i u_i^2) \mid X_n, U_{n-1} \} \]
\[ = s_n x_n^2 + 2t_n x_n \lambda + w_n \lambda^2 + k_n u_n^2 + \]
\[ \frac{1}{\rho_n} \left\{ E_k E \left[ \sum_{i=n+1}^M \frac{\rho_i}{\rho_{i+1}} \left( s_i x_i^2 + 2 t_i x_i \lambda \right) \right] \right\} + \frac{u_i \lambda^2 + k_i u_i^2}{\left( X_{n+1}, U_n \right)} \]

\[ = s_n x_n^2 + 2 t_n x_n \lambda + k_n u_n \lambda^2 + \frac{\rho_{n+1}}{\rho_n} E \left[ R_{n+1} (\lambda, U^{n+1}) \mid X_n, U_{n-1} \right] \]

\[ = \frac{\rho_{n+1}}{\rho_n} E \left[ R_{n+1} (\lambda, U^{n+1}) \mid X_n, U_{n-1} \right]. \] (11)

**Lemma 2.** The risk function \( R_n (\lambda, U^n) \) for the Bayes control policy \( U^*_n \) has from

\[ R_n (\lambda, U^*_n) = a_n x_n^2 + 2 b_n x_n x_{n-1} + c_n x_{n-1}^2 + 2 d_n x_n \lambda + 2 e_n x_{n-1} \lambda \]

\[ + f_n r_n^2 + 2 g_n r_n \lambda + h_n \lambda^2 + i_n \lambda + j_n, \] (12)

where

\[ a_n = A_n, b_n = B_n, c_n = C_n, d_n = D_n, e_n = E_n, \]

\[ f_n = \frac{1}{\rho_n} \sum_{i=n}^{M-1} \frac{P_i}{\beta_i^2}, \]

\[ g_n = \frac{1}{\rho_n} \sum_{i=n}^{M-1} Q_i \frac{1}{\beta_i} + \frac{q}{\rho_n} \sum_{i=n}^{M-1} (i-n) P_i \frac{1}{\beta_i^2}, \] (13)

\[ h_n = \frac{1}{\rho_n} z_n + 2 \frac{q}{\rho_n} \sum_{i=n+1}^{M-1} (i-n) Q_i \frac{1}{\beta_i^2} + \frac{1}{\rho_n} \sum_{i=n+1}^{M-1} (i-n) \left( q (q-\xi) + (i-n-1) q^2 P_i \right) \frac{1}{\beta_i^2}, \]

\[ i_n = \frac{q}{\rho_n} \left[ y_n + \sum_{i=n+1}^{M-1} (i-n) P_i \frac{1}{\beta_i^2} \right], \]

\[ j_n = \frac{q}{\rho_n} \left[ y_n + \sum_{i=n+1}^{M-1} (i-n) P_i \frac{1}{\beta_i^2} \right], \]

\[ P_n = \frac{2}{\rho_n} (k_n \varphi_n + a_{n+1} \varphi_{n+1} \xi_n^2), \]
\[ Q_{n} = 5_n \rho_{n+1} (a_{n+1} \gamma_{n+1} q + d_{n+1}) \tau_{n}. \]

\[ z_{n} = \frac{1}{\rho_{n}} \sum_{i=n}^{M} \rho_{w_{i}} + \frac{1}{\rho_{n}} \sum_{i=n}^{M-1} \rho_{i+1} (a_{i+1} \gamma_{i}^{2} q (q - \chi) + 2d_{i+1} q), \]

\[ y_{n} = \sum_{i=n}^{M-1} \rho_{i+1} a_{i+1} \gamma_{i}^{2}. \]

Proof. Assume that

\[ R_{n}(\lambda, U_{\beta,r}^{n*}) = a_{n} \chi_{n}^{2} + 2b_{n} x_{n} x_{n-1} + c_{n} x_{n-1}^{2} + 2d_{n} x_{n} \lambda + 2e_{n} x_{n-1} \lambda^{2} + f_{n} r_{n}^{2} + g_{n} r_{n} \lambda + h_{n} \lambda^{2} + i_{n} \lambda + j_{n} + 2l_{n} x_{n} r_{n} + 2m_{n} x_{n-1} r_{n} + o_{n} r_{n} \]  

(14)

The explicit calculations give

\[ R_{n}(\lambda, U_{\beta,r}^{M*}) = s_{M} \chi_{M}^{2} + 2l_{M} x_{M} \lambda + w_{M} \lambda^{2}. \]  

(15)

Let us assume for the induction that \( R_{n+1}(\lambda, U_{\beta,r}^{n+1*}) \) has the form given by (14). By (11), taking (7) into account we obtain that \( R_{n}(\lambda, U_{\beta,r}^{n*}) \) satisfies (14) with the coefficients satisfying recurrent equations, which are equivalent to (13) by (15) and (10).

4. A Minimax Control

From Lemma 2 we obtain an explicit form of the risk function for the Bayes control policy. It can be written also as

\[ R(\lambda, U_{\beta,r}^{*}) = h_{o}(\beta) \lambda^{2} + Z(\beta, r) \lambda + T(\beta, r), \]  

(16)

where

\[ h_{o}(\beta) = h_{o}, \]

\[ Z(\beta, r) = 2d_{o} x_{o} + 2e_{o} x_{o-1} + 2g_{o} r + i_{o}, \]  

(17)

\[ T(\beta, r) = a_{o} x_{o}^{2} + 2b_{o} x_{o} x_{o-1} + c_{o} x_{o-1}^{2} + f_{o} r^{2} + j_{o}. \]

The parameter space \( \Delta \) is equal to \( R^{+} \) for the Poisson, gamma, negative binomial distributions and to \( R \) for the normal, GEHS distributions. The form of the risk function and the set \( \Delta \) for these five distributions lead to the conclusion that when we have no restrictions on \( \lambda \), the minimax
control rarely exists. It is a consequence of unboundness of $R(\lambda, U^r_{\beta, r})$ on $\Lambda$.

The set $\Lambda$ for the binomial distribution is equal to (0,1). It gives possibility to search the minimax control for such disturbances.

Further considerations are based on the following:

**Lemma 3** (see [8], [9]). Suppose that there is a sequence $\{p_k\}, k = 1, 2, \ldots$ of prior distributions belonging to $\Gamma_o$ for which the corresponding sequence $\{U^o_{\pi_k}\}, k = 1, 2, \ldots$ of the Bayes control policies satisfies the condition

$$\lim_{k \to \infty} r(p_k, U^o_{\pi_k}) = c$$

and there is a control policy $\hat{u} \in \mathcal{D}$ such that $r(p, \hat{u}) \leq c$ for each $p \in \Gamma_o$ then $\hat{u}$ is a $\Gamma_o$-minimax control policy.

From Lemma 3 we have among other things that the Bayes control policy $\hat{u}$, for which $r(p, \hat{u}) = \text{const}$, for each $p \in \Gamma_o$, is $\Gamma_o$-minimax policy.

Let us define two control policies $U^+_m$ and $U^-_{\tau, m}$, which can be obtained as a limit of Bayes control policies with probability one.

Denote $U^+_m = (u^+_0, u^+_1, \ldots, u^+_M)$ the control policy defined by

$$u^+_0 = 0$$
$$u^+_n = c_{\pi_n} X_n + c_{\pi_{n-1}} + \tau_n m, n = 0, 1, \ldots, M - 1.$$  

(18)

$U^+_m$ may be treated as the a.s. limit

$$U^+_m = \lim_{\beta \to \infty} U^*_{\beta, r} = \lim_{\beta \to \infty} \lim_{\tau \to m} U^*_{\beta, r},$$

where $(\beta, r) \in S^2$.

From (16) and (17), we have

$$R(\lambda, U^+_m) = \lim_{\beta \to \infty} R(\lambda, U^*_{\beta, r}) = z_o \lambda^2 + (2d_o x_o + 2e_o x_{o-1} + \gamma y_o) \lambda$$

$$+ 2m \sum_{i=0}^{M-1} Q_i + m^2 \sum_{i=0}^{M-1} P_i + a_o x_o^2 + 2b_o x_o x_{o-1} + c_o x_{o-1}^2 + \nu y_o y_o.$$  

(19)
Denote $U_{\xi,m} = (u_0, u_1, \ldots, u_M)$ the control policy defined by:

$$
u_M = 0,$$

$$u_n = -\xi_n \xi_n + \sum_{i=0}^{n-1} r_i \xi_i + \tau_n (m),$$

where $\xi_n = \xi + nq; r_n = \sum_{i=0}^{n-1} \nu_i + m\xi; u_0 = -\xi_0 + \nu_0 + \tau_0 (m).$

$U_{\xi,m}$ can be obtained as a.s. limit of the Bayes control policies

$$U_{\xi,m} = \lim_{\beta \to \xi, \beta \to \xi} U^*_\beta.$$

By the form of $R(\lambda, U^*_{\beta,r})$ we have

$$R(\lambda, U_{\xi,m}) = \lim_{\beta \to \xi, \beta \to \xi} R(\lambda, U^*_{\beta,r}).$$

Notice also that

$$r(\pi, U^*_{\beta,r}) = E_\pi R(\lambda, U^*_{\beta,r}) h_0(\beta) E_\pi \lambda^2 + Z(\beta, r) E_\pi \lambda + T(\beta, r).$$

Now, the minimax controls for the binomial disturbances will be determined.

**Theorem 1.** Let the SOLS be given by (1) and the cost of the control by (5) with the disturbances having the binomial distribution with parameters $q$ and $\lambda$, then

**A.** If $\beta^* > 0$ is such that $h_0(\beta^*) = 0$ and

(i) there is $r^* \xi(0, \beta^*)$ such that $Z(\beta^*, r) = 0$, then the policy $U^*_{\beta^*, r}$ is the minimax policy (m.p.);

(ii) $Z(\beta^*, r) > 0$ for all $r \in (0, \beta^*)$, then $U_{\beta^*, 0}$ is the m.p.;

(iii) $Z(\beta^*, r) > 0$ for all $r \in (0, \beta^*)$, then $U^*_{\beta^*, 1}$ is the m.p.
B. Let \( h_\beta(\beta) < 0 \) for all \( \beta > 0 \). Define

\[
\psi(m) = (z_0 - \sum_{i=0}^{M-1} P_i) m^2 + 2d_0 x_0 + 2e_0 x_{-1} + qy_0 \, m \\
+ a_0 x_0^2 + 2b_0 x_0 x_{-1} + c_0 x_{-1}^2
\]  
(23)

Let \( m^* \) be such that

\[
\psi(m^*) = \sup_{m \in [0,1]} \psi(m),
\]  
(24)

then \( U_{m^*}^* \) is the m.p. in this case.

C. Let \( h_\beta(\beta) > 0 \) for all \( \beta > 0 \). Define

\[
\psi(m) = -P_0 m^2 + (z_0 - \sum_{i=1}^{M-1} P_i + 2d_0 x_0 + 2e_0 x_{-1} + qy_0) \, m \\
+ a_0 x_0^2 + 2b_0 x_0 x_{-1} + c_0 x_{-1}^2
\]  
(25)

Let \( m \) be such that

\[
\psi(m) = \sup_{m \in [0,1]} \psi(m),
\]  
(26)

then \( U_{0,m}^- \) is the m.p..

Proof. First of all let us recall that for the binomial distribution we have

by the assumptions (2) and (3), \( E r_n^2 = q(q-1) \lambda^2 + q\lambda \).

In the case \( A(i) \) the policy \( U_{\beta^* r^*}^* \) is of the constant risk, i.e., \( r(\pi, U_{\beta^* r^*}^*) = \text{const.} \) Independently of \( \pi \), then it is the m.p.

In the case \( A(ii) \), by (22) and (21) we have

\[
\lim_{\beta \to \beta^*} r(\pi_{\beta} r, U_{\beta,r}^*) = T(\beta^*,0^*)
\]

\[
\lim_{\beta \to 0} \pi_{\beta} = 0
\]

and from (16), (17), (21), we obtain

\[
R(\lambda, U_{\beta^*,0}^-) = Z(\beta^*,0^+) \lambda \leq T(\beta^*,0^+) \leq T(\beta^*,0^+).
\]

Therefore, by Lemma 3, \( U_{\beta^*,0}^- \) is the m.p.
The case \( A (iii) \) can be proved similarly.

Suppose that the assumptions of \( B \) are satisfied. The condition
\[
h_o(\beta) < 0, \beta > 0, \text{ implies } z_o \leqslant 0 \text{ and } z_o - \sum_{i=0}^{M-1} P_i \leqslant 0. \text{ Notice that}
\]
\[
Q_i = -P_i. \text{ We have}
\]
\[
R(\lambda, U^*_m) = (z_o - \sum_{i=0}^{M-1} P_i) \lambda^2 + (2d_o x_o + 2c_o x_{-1} + q y_o) \lambda
\]
\[
+ a_o x_o^2 + 2b_o x_o x_{-1} + c_o x_{-1}^2 + (\lambda - m^*)^2 \sum_{i=0}^{M-1} P_i
\]
and
\[
\lim_{\beta \to \infty} r(\pi_{\beta, r}, U^*_m) = \phi(m^*) \tag{23}
\]
\[
\frac{r}{\beta} \to m^*
\]
where \( m^* \) is defined by (24). Needless to say that such \( m^* \) always exists. Since \( z_o - \sum_{i=0}^{M-1} P_i \leqslant 0 \), then \( R(\lambda, U^*_m) \) is the concave, quadratic function of \( \lambda \) and the value of the maximum in \((0, 1)\) is attained at
\[
\lambda = m^* \tag{25}
\]
This maximum is equal to \( \phi(m^*) \). Therefore \( R(\lambda, U^*_m) \leqslant \phi(m^*) \)
for each \( \lambda \in (0, 1) \). Using Lemma 3 we obtain that \( U^*_m \) is the m.p..

Suppose that the assumptions of \( C \) are satisfied. The condition \( h_o(\beta) > 0 \) for \( \beta > 0 \) implies \( z_o - \sum_{i=1}^{M-1} P_i \leqslant 0 \text{ and } \sum_{i=1}^{M-1} P_i > 0. \text{ Since } \lambda^2 \leqslant \lambda \text{ for } \lambda \in (0, 1) \), we have
\[
R(\lambda, U^*_{0, m}) = (z_o - \sum_{i=1}^{M-1} P_i \frac{M-1}{iq}) \lambda^2
\]
\[
+ (2d_o x_o + 2c_o x_{-1} + q y_o + \sum_{i=1}^{M-1} P_i \frac{M-1}{iq}) \lambda + m^* P_o - 2P_o m \lambda
\]
\[
+ a_o x_o^2 + 2b_o x_o x_{-1} + c_o x_{-1}^2 \leqslant -P_o \lambda^2 + (z_o - \sum_{i=1}^{M-1} P_i
\]
\[
+ 2d_o x_o + 2c_o x_{-1} + q y_o) \lambda
\]
\[
+ a_o x_o^2 + 2b_o x_o x_{-1} + c_o x_{-1}^2 + P_o (\lambda - m)^2.
\]
We also have
\[
\lim_{\beta \to 0^+} \lim_{r \to m} r (\pi_{\beta, r}, U_{\beta, r}^*) = \Psi(m).
\]

Since \( R(\lambda, U_{0, m}^-) \) is bounded by the linear function which attains the maximum at \( \lambda = m \) and the value of the maximum is equal to \( \Psi(m) \), we have
\[
R(\lambda, U_{0, m}^-) \leq \Psi(m).
\]

By Lemma 3 the policy \( U_{0, m}^- \) is the m.p. in this case. These complete the proof of Theorem 1.

When restrictions on the set of prior distributions are given, the minimax controls can be obtained for another class of disturbances.

Let \( m_1 \in (0, 1) \) and define \( \Gamma_1 = \{ \pi \in \Gamma : E_{\pi} \lambda = m_1 \} \). Let us assume that the disturbances have the binomial distribution. Denote \( S_1 = \{ (\beta, r) \in S^2 : E_{\pi_{\beta, r}} \lambda = m_1 \} \).

**Theorem 2.** Let the SOLS be given by (1) and the cost function by (5) with disturbances having the binomial distribution with the parameters \( q \) and \( \lambda \). Assume that the parameter \( \lambda \) is a random variable with a distribution \( \pi \in \Gamma_1 \), where \( m_1 \) is given.

(i) If there is \( \beta^* > 0 \) such that \( h_0(\beta^*) = 0 \), then \( U^* \) is the \( \Gamma_1 \)-minimax \( \pi_{\beta^*, m_1}^* \) policy (\( \Gamma_1 \)-m.p.);

(ii) If \( h_0(\beta) > 0 \) for all \( \beta > 0 \), then \( U_{\beta, m_1}^- = \Gamma_1 \)-m.p. ;

(iii) If \( h_0(\beta) < 0 \) for all \( \beta > 0 \), then \( U_{m_1}^+ = \Gamma_1 \)-m.p.

Let us assume that \( \pi \in \Gamma_1 = \{ \pi \in \Gamma : E_{\pi} \lambda = m_1, m_1 \in (\Lambda)^2 \} \). Denote \( S_1 = (\beta, r) \in S^2 : E_{\pi_{\beta, r}} \lambda^2 = m_2 \).

**Theorem 3.** Suppose that the SOLS is given by (1) and the cost of control by (5). For the disturbances belonging to the exponential class with the quadratic variance function, when the prior distribution of the parameter \( \lambda \) belongs to the class \( \Gamma_1 \), we have
A. When there exists $(\beta^*, r^*) \in S_2$ such that $Z(\beta^*, r^*) = 0$ then the Bayes control policy $U_{\beta^*, r^*}^*$ given by (8) and (9) is the $\Gamma_2$-m.p.

B. When $Z(\beta, r) > 0$ for each $(\beta, r) \in S_2$, then the policy $U_{\sqrt{m_2}}^+$ defined by (18) is the $\Gamma_2$-m.p.

C. When $Z(\beta, r) < 0$ for each $(\beta, r) \in S_2$ then

(i) for the normal and GEHS distribution—the policy $U_{-\sqrt{m_2}}^+$,

(ii) for the Poisson distribution—the policy $U_{0, 0}^-$ defined by (20);

(iii) for the gamma and the negative binomial distribution—the policy $U_{1, 0}^-$

(iv) for the binomial distribution—the policy $U_{0, m_2}^-$ are the $\Gamma_2$-m.p.

Let $\Gamma_{1, 2} = \{ \pi \in \Gamma : E_\pi \lambda = m_1, E_\pi \lambda^2 = m_2, m_1 \in \Lambda, m_2 \in (\Lambda)^2, m_1, m_2 \geq 0 \}$ and $S_{1, 2} = \{ (\beta, r) \in S_2 : E_{\pi_{\beta, r}} \lambda = m_1, E_{\pi_{\beta, r}} \lambda^2 = m_2 \}$. We assume that $m_1$ and $m_2$ are such that $\Gamma_{1, 2} \neq \emptyset$ and $S_{1, 2} \neq \emptyset$.

**Theorem 4.** Assume that the SOLS is given by (1) and the cost function of control by (5). For the disturbances belonging to the exponential class with the quadratic variance function, when the prior distribution of the parameter $\lambda$ belongs to the class $\Gamma_{1, 2}$, the $\Gamma_{1, 2}$-m.p. is $U_{\beta^*, r^*}^*$, where $(\beta^*, r^*) \in S_{1, 2}$.

The proofs of Theorems 2, 3, and 4 are based on Lemma 3. These are very similar to the proof of Theorem 1 and are omitted.

5. Final Remarks

The restriction to the SOLS only allowed to obtain both the minimax controls and the sufficient conditions for the existence of them in a clear form. For the $m$ order linear system described by

$$x_{n+1} = \sum_{i=1}^{m} a_{i, n} x_{n-m+i} + \delta_n u_n + \gamma_n v_n, x_0 \in R, u_n \in R$$

$$n = 0, 1, \ldots, N-1 : x_0, x_1, \ldots, x_{N-m+1} — \text{given}$$

and the cost function of control given by (5), when the disturbances have
the properties described in Sections 2 and 3, the Bayes control can be obtained from the results of the paper [5]. At instant n this control is a linear function of the last m states and parameters of the posterior distribution. It can be shown that the risk function is the quadratic function of the parameter λ given by (20) with the coefficients \( b_n (\beta) \), \( Z (\beta, r) \) and \( I(\beta, r) \) having a different form than in the SOLS case. The exact form of them can be calculated analogously, as in the second order case, recursively. Defining controls similarly as in (22) and (24), the sufficient conditions for the existence of minimax controls and their form can be obtained.

References


