BAYES CONTROL
FOR A MULTIDIMENSIONAL STOCHASTIC SYSTEM

The paper deals with a discrete time multidimensional linear stochastic system with a random horizon of a control. The disturbances in the system are given by a random vector with components belonging to an exponential family with a parameter \( \lambda \). The cost of the control is the quadratic form of states, the parameter \( \lambda \) of disturbances and controls. The horizon of a control is a random variable with the known distribution, independent of the disturbances. For a conjugate prior distribution of the parameter \( \lambda \) the Bayes control is obtained.


1. INTRODUCTION

Let us consider an \( m \)-dimensional linear system defined by the equation

\[
\tilde{x}_{n+1} = a_n \tilde{x}_n + b_n \tilde{u}_n + c_n \tilde{v}_n; \quad n = 0, 1, \ldots, N - 1
\]

where \( \tilde{x}_n \) is the state of the system, \( \tilde{u}_n \) is the control, \( \tilde{v}_n \) is the disturbance at a time \( n \); \( \tilde{x}_n, \tilde{u}_n, \tilde{v}_n \), are the \( m \)-dimensional vectors; \( a_n, b_n, c_n \) are the given \( m \times m \) matrices. The horizon of the control \( N \) is a random variable, independent of \( \tilde{v}_0, \tilde{v}_1, \ldots \), with the known distribution given by

\[
P\{N = k\} = p_k, \quad k = 0, 1, \ldots, M, \quad \sum_{k=0}^{M} p_k = 1, \quad p_M \neq 0.
\]

The disturbances \( \tilde{v}_n \) considered in the paper belong to a \( k \)-dimensional linear space \( 1 \leq k \leq m \) \( V \subset R^m \) and the matrices \( c_n \) are such that for \( \tilde{z} \in c_n(V) \)

\[\{\tilde{z} \in R^m: \exists v_n \in c_n \tilde{v} = \tilde{z}\}\] linear equations \( c_n \tilde{v}_n = \tilde{z} \) have exactly one solution \( \tilde{v}_n \),

\( n = 0, 1, \ldots, M \). Without loss of generality we can assume that \( \tilde{v}_n \) has the form

\( \tilde{v}_n = (v_n^1, v_n^2, \ldots, v_n^k, 0, \ldots, 0)^T \). The vectors \( \tilde{v}_n, \quad n = 0, 1, \ldots, M \) are independent,
identically distributed random vectors with the components $v_i^n, i = 1, 2, \ldots, k$, having the distribution belonging to an exponential family (they may also belong to different families for different $i$), dependent on the unknown parameter $\lambda_i, i = 1, 2, \ldots, k$.

The following notations are used: $X_n = (\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_n); \ U_n = (\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_n); \ U^n = (\bar{u}_1, \bar{u}_{n+1}, \ldots, \bar{u}_M); \ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k, 0, \ldots, 0)^T$. For the convenience $U_M$ will be denoted by $U$ and called a control policy.

It is assumed that at the moment $n$ both $X_n$ and $U_{n-1}$ are given and the control $\bar{u}_n$ is based on these information. Therefore before any data are obtained the control $\bar{u}_n$ is a random vector determined by the random disturbances $\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_{n-1}$.

Let us assume that the cost of a control for the given control policy $U$ (the loss function) is given by

$$L(U, X_N) = \sum_{i=0}^{N} (\bar{y}_i^T s_i \bar{y}_i + \bar{u}_i^T k_i \bar{u}_i)$$

where $s_i, i = 0, 1, \ldots, M,$ are the non-negatively definite $2m \times 2m$-matrices, $k_i, i = 0, 1, \ldots, M,$ are the non-negatively definite $m \times m$-matrices and $\bar{y}_i = (\bar{x}_i^T, \lambda^T)^T, i = 0, 1, \ldots, M,$ are the $2m$-dimensional vectors.

The risk connected with the control policy $U$, when the parameter $\lambda$ is given, is defined as follows

$$R(\lambda, U) = E_pE_{\lambda}[L(U, X_N)] = E_pE_{\lambda}\{\sum_{i=0}^{N} (\bar{y}_i^T s_i \bar{y}_i + \bar{u}_i^T k_i \bar{u}_i)|X_0}\}.$$

For the prior distribution $\pi$ of the parameter $\lambda$ the expected risk $r$, associated with $\pi$ and the control policy $U$, is

$$r(\pi, U) = E_\pi [R(\lambda, U)] = E_pE_{\pi}\{\sum_{i=0}^{N} (\bar{y}_i^T s_i \bar{y}_i + \bar{u}_i^T k_i \bar{u}_i)|X_0}\}$$

where $E_\pi, E_{\lambda}$ denote the expectations with respect to the distributions of $N$ and random vectors $\bar{v}_0, \bar{v}_1, \ldots$ (when $\lambda$ is the parameter), $E_\pi$ and $E$ with respect to the distribution $\pi$ and to the joint distribution $\bar{v}_n$ and $\bar{\lambda}$, respectively.

Let the initial state $\bar{x}_0$ and the distribution $\pi$ of the parameter $\lambda$ be given. A control policy $U^*$ is called the Bayes policy when

$$r(\pi, U^*) = \inf_{U \in \mathcal{D}_\pi} r(\pi, U)$$

where $\mathcal{D}_\pi$ is the class of the control policies $U$ for which there exists $r(\pi, U)$.

Suppose that the disturbances $v^n_i, n = 0, 1, \ldots$ belong to an exponential family with a natural parameter $\lambda_i, i = 1, 2, \ldots, k$, respectively. To this family there belong distributions such as the binominal, normal, the gamma and Poisson ones, which are very frequently considered in practice. Section 2 provides some remarks.
on disturbances. The problem of determining the Bayes control for the conjugate prior distribution \( \pi \) of the parameter \( \bar{\lambda} \) is solved in Section 3. The results of Section 3 are used in Section 4 to obtain the Bayes control for a second order discrete linear system.

The problem of determining the Bayes control of the stochastic systems for disturbances belonging to an exponential family for one dimensional systems was considered in [1], [8], [9].

2. FILTERING

We assume that the disturbances have the distribution belonging to an exponential family [4], [5], [7]. More precisely we assume that \( \bar{v}_n \) has the distribution with the density \( p(\bar{v}_n, \bar{\lambda}) \) with respect to some \( \sigma \)-finite measure \( \mu \) on \( R^m \) of the form:

\[
p(\bar{v}_n, \bar{\lambda}) = \prod_{i=1}^{k} p(v^i_n, \lambda_i)
\]

where

\[
p(v^i_n, \lambda_i) = S_i(v^i_n, q_i) \exp \{ q_i A_i(\lambda_i) + v^i_n B_i(\lambda_i) \}, \quad q_i \in Q^*_i, \quad v^i_n \in V^*_i, \quad i = 1, 2, \ldots, k,
\]

i.e. we suppose that the distribution of the components of \( \bar{v}_n \) belongs to an exponential family or is equal to \( 0 (Q^*_i, V^*_i, A_i(\cdot), B_i(\cdot)) \) are defined in [8].

We take the parametrization in such a way that

\[
E_{\bar{\lambda}}(v^i_n) = -q_i A^i_i(\lambda_i)/B^i_i(\lambda_i) = q_i \lambda_i, \quad \text{for some } q_i > 0
\]

and it is assumed that

\[
E_{\bar{\lambda}}(v^i_n)^2 = q_{1,i} \lambda_i^2 + q_{2,i} \lambda_i + q_{3,i}
\]

where \( q_{1,i}, q_{2,i}, q_{3,i} \) are some constants.

Let \( \bar{v}_n \) have the density given by (3) when \( \bar{\lambda} \) is the only unknown parameter. Let us suppose that the prior distribution \( \pi \) of \( \bar{\lambda} \) has the distribution conjugate to (3), i.e. the density of \( \pi \) has the form:

\[
g(\bar{\lambda}; \bar{\beta}, \bar{r}) = \prod_{i=1}^{k} g_i(\lambda_i; \beta^i, r^i)
\]

where

\[
g_i(\lambda_i; \beta^i, r^i) = C_i(\beta^i, r^i) B^i_i(\lambda_i) \exp \{ \beta^i A^i_i(\lambda_i) + r^i B^i_i(\lambda_i) \}, \quad (\beta^i, r^i) \in S^i
\]

and \( \bar{\lambda} \in A = B^{-1}(A_0), A_0 \) is the natural parameter space.

To determine the Bayes control the posterior density for \( \bar{\lambda} \) must be obtained.
after any new observation. It is possible if the matrices $c_n$ in (1) are such that the equations

$$c_n \bar{v}_n = \bar{x}_{n+1} - a_n \bar{x}_n - b_n \bar{u}_n, \quad \bar{x}_0 \quad \text{given} \quad (5)$$

have the unique solutions $\bar{v}_n$, $n = 0, 1, \ldots, N - 1$. In these cases the posterior density $f(\lambda|X_n, U_{n-1})$ of the parameter $\lambda$, after having observed $X_n$ and choosen $U_{n-1}$, has the same form as (4) i.e.

$$f(\lambda|X_n, U_{n-1}) = f(\lambda|V_{n-1}) = g(\lambda; \bar{\beta}_n, \bar{r}_n)$$

where $V_{n-1} = (\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_{n-1})$ and $\bar{\beta}_n$, $\bar{r}_n$ have the form:

$$\bar{\beta}_n = \bar{\beta}_{n-1} + \bar{q} = \bar{\beta} + n\bar{q},$$

$$\bar{r}_n = \bar{r} + \sum_{j=0}^{n-1} \bar{v}_j; \quad \bar{r}_0 = \bar{r}.$$  

From [5] it follows that in order to obtain the considered exponential class of the distributions with the variance being the quadratic function of the mean, it suffices to take account of the following families of the distributions: the binomial, gamma, generalized exponential hyperbolic secant (GEHS), negative binomial, normal and the Poisson distributions.

For all these distributions (when (5) has the unique solution) we have

$$E(\lambda_i|X_n, U_{n-1}) = T^{n,i}_1 \bar{r}^i_n = r^i_n / \beta^i_n, \quad (6)$$

$$E(\lambda_i^2|X_n, U_{n-1}) = T^{n,i}_1 (r^i_n)^2 + T^{n,i}_2 r^i_n + T^{n,i}_3$$

where $T^{n,i}_j$, $j = 1, 2, 3$, $n = 0, 1, \ldots, M$ are constants dependent on $\beta^i$, $i = 1, 2, \ldots, k$.

When $X_n$ and $U_{n-1}$ are given, the conditional distribution of the random variable $\bar{v}_n$ has the density of the form (see [9]):

$$h(\bar{v}|X_n, U_{n-1}) = \prod_{i=1}^{k} h_i(\bar{v}|X_n, U_{n-1})$$

where

$$h_i(\bar{v}|X_n, U_{n-1}) = S_i(\bar{v}, q_i) \frac{C_i(\beta^i_n, r^i_n)}{C_i(\beta^i_{n+1}, r^i_{n+1})}, \quad i = 1, 2, \ldots, k, \quad n = 0, 1, \ldots, M - 1.$$  

We have

$$E(\bar{v}^i_n|X_n, U_{n-1}) = Q^{n,i}_1 r^i_n,$$

$$E[(\bar{v}^i_n)^2|X_n, U_{n-1}] = Q^{n,i}_1 (r^i_n)^2 + Q^{n,i}_2 r^i_n + Q^{n,i}_3$$

for some constants $Q^{n,i}_j$, $j = 1, 2, 3$, $n = 0, 1, \ldots, M$, dependent on the parameter $\beta^i$, $i = 1, 2, \ldots, k$. 
Bayes control for a stochastic system

The constants $T_{j}^{n_i}, Q_{j}^{n_i}, j = 1, 2, 3, i = 1, 2, \ldots, k$ can be found in [6] and [10] for the binomial, gamma, normal, negative binomial and the Poisson distributions. For the GEHS distribution the constants are given in Appendix.

In the paper we use the following notations: $ar{\xi} = (\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_m)^T$ and

$$
\bar{\xi} = \begin{pmatrix}
\bar{\xi}_1 & 0 & \cdots & 0 \\
0 & \bar{\xi}_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \bar{\xi}_m
\end{pmatrix},
$$

if $A = (a_{ij})_{m \times m}$, then

$$
diag A = \begin{pmatrix}
a_{11} & 0 & \cdots & 0 \\
0 & a_{22} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & a_{mm}
\end{pmatrix},
$$

and $\bar{e} = (e_1, e_2, \ldots, e_m)^T$ where $e_i = 1$ for $i = 1, 2, \ldots, k$ and $e_i = 0$ for $i = k+1, k+2, \ldots, m$.

3. THE BAYES CONTROL

Suppose the initial state $\bar{x}_0$ is given, the disturbances have the distribution with the density given by (3) and the prior distribution $\pi$ of the parameter $\bar{x}$ is given by (4). Let the distribution of the random horizon $N$ be given by (2). Consider the problem of the Bayes control for the system (1) with the starting point at the moment $n$, when $X_n, U_{n-1}$ are given. The expected risk is then:

$$
r_n = r_n(\pi, U^n) = E_p \{ E \left[ \sum_{i=n}^{N} (\bar{y}_i^T s_i \bar{y}_i + \bar{u}_i^T k_i \bar{u}_i) \left| X_n, U_{n-1} \right. \right] \} | N \geq n \}.
$$

(8)

Let us denote

$$
\varphi_k = \sum_{i=k}^{M} p_i.
$$

We have

$$
r_n = \sum_{k=n}^{M} E \left[ \sum_{i=n}^{k} (\bar{y}_i^T s_i \bar{y}_i + \bar{u}_i^T k_i \bar{u}_i) \left| X_n, U_{n-1} \right. \right] \frac{p_k}{\varphi_n}
$$

$$
= E \left[ \sum_{i=n}^{M} \frac{\varphi_i}{\varphi_n} (\bar{y}_i^T s_i \bar{y}_i + \bar{u}_i^T k_i \bar{u}_i) \left| X_n, U_{n-1} \right. \right].
$$
The Bayes risk for the above truncated problem is the following

\[ W_n = \inf_{U^n} r_n(\pi, U^n) = r_n(\pi, U^n), \]

where \( U^n = (\bar{u}^*_n, \bar{u}^*_{n+1}, \ldots, \bar{u}^*_M) \) is the Bayes policy, \( \bar{u}^*_i, \ i = n, n+1, \ldots, M \) are called the Bayes controls. Obviously, \( r_0(\pi, U^0) = r(\pi, U) \) and \( W_0 = r(\pi, U^*) \).

To solve the Bayes control problem we derive the Bayes control \( \bar{u}^*_n \) for \( n = M, M-1, \ldots, 1, 0 \) recursively. Then \( U^{0*} \) is the solution of the problem.

From the Bellman’s dynamic programming optimality principle we obtain the following

**Lemma**

The Bayes risk \( W_n, n = 0, 1, \ldots, M \) in the considered problem satisfies the equation

\[
W_n = \min_{u_n} \left\{ \left( \bar{x}_n^T \bar{s}_n \bar{x}_n + 2 \bar{r}_n^T T^n \bar{s}_n^3 \bar{x}_n + \bar{r}_n^T \left[ T^n \bar{s}_n^2 T^n + \left( T^n - (T^n)^2 \right) \text{diag} \bar{s}_n^2 \right] \bar{r}_n 
+ (T_2^n)^T \text{diag} \bar{s}_n^2 \bar{r}_n + \bar{e}^T \text{diag} \bar{s}_n^2 T_3^n \bar{r}_n + \bar{u}_n^T k_n \bar{u}_n + \frac{\phi_{n+1}}{\phi_n} E(W_{n+1}|X_n, U_{n-1}) \right\}
\]  

(9)

with the condition

\[
W_M = \bar{x}_M^T s_M \bar{x}_M + 2 \bar{r}_M^T T^M s_M \bar{x}_M + \bar{r}_M^T \left[ T^M s_M^2 T^M + \left( T^M - (T^M)^2 \right) \text{diag} s_M^2 \right] \bar{r}_M 
+ (T_2^M)^T \text{diag} s_M^2 \bar{r}_M + \bar{e}^T \text{diag} s_M^2 T_3^M
\]

(10)

where \( s_i = \begin{pmatrix} s_i^1 & (s_i^3)^T \end{pmatrix} \).

**Proof**

For \( n = M \) we have

\[
W_M = \min_{u_M} E(\bar{y}_M^T s_M \bar{y}_M + \bar{u}_M^T k_M \bar{u}_M|X_M, U_{M-1}) = E(\bar{y}_M^T s_M \bar{y}_M|X_M, U_{M-1}),
\]

then \( \bar{u}_M^* = 0 \).

If \( n = 0, 1, \ldots, M-1 \), then by (8) and the property of the conditional expectation:

\[
W_n = \min_{U^n} r_n = \min_{U^n} \left\{ \min_{U^n} E \left[ (\bar{y}_n^T s_n \bar{y}_n + \bar{u}_n^T k_n \bar{u}_n)|X_n, U_{n-1} \right] 
+ E \left[ \sum_{i=n+1}^{M} \frac{\phi_i}{\phi_n} (\bar{y}_i^T s_i \bar{y}_i + \bar{u}_i^T k_i \bar{u}_i)|X_n, U_{n-1} \right] \right\}
\]

\[
= \min_{U_n} \left\{ \bar{u}_n^T k_n \bar{u}_n + E(\bar{y}_n^T s_n \bar{y}_n|X_n, U_{n-1}) 
+ \min_{U^{n+1}} \left( \frac{\phi_{n+1}}{\phi_n} \sum_{i=n+1}^{M} \frac{\phi_i}{\phi_{n+1}} (\bar{y}_i^T s_i \bar{y}_i + \bar{u}_i^T k_i \bar{u}_i)|X_{n+1}, U_n \right) \right\}.
\]
Hence

$$W_n = \min_{\bar{u}_n} \left\{ \bar{u}_n^T k_n \bar{u}_n + E(T_n^T s_n \bar{x}_n | X_n, U_{n-1}) + \frac{\varphi_{n+1}}{\varphi_n} E(W_{n+1} | X_n, U_{n-1}) \right\}. \quad (12)$$

Now we calculate $E(T_n^T s_n \bar{x}_n | X_n, U_{n-1})$ for $n = 0, 1, \ldots, M$. From (6) we obtain

$$E(T_n^T s_n \bar{x}_n | X_n, U_{n-1}) = \bar{x}_n^T s_n \bar{x}_n,$$

$$E(T_n^T s_n \bar{x}_n | X_n, U_{n-1}) = \bar{r}_n^T T_n^T s_n \bar{x}_n,$$

$$E(T_n^T s_n \bar{x}_n | X_n, U_{n-1}) = (T_n^T \bar{r}_n) (s_n^2 - \text{diag } s_n^2) (T_n^T \bar{r}_n)$$

$$+ \bar{r}_n^T T_1^T \text{diag } s_n^2 \bar{r}_n + (T_2^n)^T \text{diag } s_n^2 \bar{r}_n + \bar{e}^T \text{diag } s_n^2 T_3^n$$

$$= \bar{r}_n^T [T_n^T s_n T_n^T + (T_1^n - (T_n^T)^2) \text{diag } s_n^2] \bar{r}_n + (T_2^n)^T \text{diag } s_n^2 \bar{r}_n + \bar{e}^T \text{diag } s_n^2 T_3^n.$$

Using these relations and taking into account (1), we can write (11) and (12) in the form (9) and (10), respectively. The lemma is thus proved.

This lemma and the inductive arguments are used to prove that the Bayes control $\bar{u}_n^*$ satisfies the equation:

$$2k_n \bar{u}_n^* + \text{grad}_{\bar{u}_n} \frac{\varphi_{n+1}}{\varphi_n} E(W_{n+1} | X_n, U_{n-1}) \bigg|_{\bar{u}_n = \bar{u}_n^*} = 0$$

and which, by (1), can be presented in the form

$$2k_n \bar{u}_n^* + \frac{\varphi_{n+1}}{\varphi_n} b_n^T E(\text{grad}_{\bar{x}_n+1} W_{n+1} | X_n, U_{n-1}) \bigg|_{\bar{u}_n = \bar{u}_n^*} = 0. \quad (14)$$

We shall show that the bayes risk $W_n$ has the form:

$$W_n = \bar{x}_n^T A_n \bar{x}_n + 2\bar{r}_n^T B_n \bar{x}_n + \bar{r}_n^T C_n \bar{r}_n + \bar{D}_n^T \bar{r}_n + E_n,$$

where $A_n, B_n, C_n$ are the $m \times m$-matrices, $\bar{D}_n$ is the vector with $m$ components and $E_n$ is the constant.

It holds for $n = M$ and

$$A_M = s_M^1, \quad B_M = T_M s_M^2, \quad C_M = T_M^2 + (T_M^2)^2 \text{diag } s_M^2,$$

$$\bar{D}_M = (T_2^M)^T \text{diag } s_M^2, \quad E_M = \bar{e}^T \text{diag } s_M^2 T_3^M. \quad (15)$$

The relations (15) and (14) lead to the equation:

$$k_n \bar{u}_n^* + \frac{\varphi_{n+1}}{\varphi_n} b_n^T [A_{n+1} (a_n \bar{x}_n + b_n \bar{u}_n^*) + A_{n+1} c_n Q^n \bar{r}_n + B_{n+1}^T (e + Q^n) \bar{r}_n] = 0$$

or

$$\left( k_n + \frac{\varphi_{n+1}}{\varphi_n} b_n^T A_{n+1} b_n \right) \bar{u}_n^*$$

$$= -\frac{\varphi_{n+1}}{\varphi_n} b_n^T A_{n+1} a_n \bar{x}_n - \frac{\varphi_{n+1}}{\varphi_n} b_n^T [A_{n+1} c_n Q^n + B_{n+1}^T (e + Q^n)] \bar{r}_n. \quad (17)$$
Assume that the equality (17) has a unique solution \( \tilde{u}_n^* \) (it suffices to assume, for example, that \( k_n \) is a non-singular matrix), then the Bayes control \( \tilde{u}_n^* \) has the form

\[
\tilde{u}_n^* = -\zeta_n \tilde{x}_n - \eta_n \tilde{r}_n
\]

where

\[
\zeta_n = \frac{\varphi_{n+1}}{\varphi_n} \left( k_n + \frac{\varphi_{n+1}}{\varphi_n} b_n^T A_{n+1} b_n \right)^+ b_n^T A_{n+1} a_n,
\]

\[
\eta_n = \frac{\varphi_{n+1}}{\varphi_n} \left( k_n + \frac{\varphi_{n+1}}{\varphi_n} b_n^T A_{n+1} b_n \right)^+ b_n^T [A_{n+1} c_n Q^n + B_{n+1}^T (e + Q^n)]
\]

and

\[
\left( k_n + \frac{\varphi_{n+1}}{\varphi_n} b_n^T A_{n+1} b_n \right)^+
\]

is the Moore–Penrose pseudoinverse matrix [6] of the matrix

\[
\left( k_n + \frac{\varphi_{n+1}}{\varphi_n} b_n^T A_{n+1} b_n \right).
\]

From (15) and (7) we have

\[
E(W_{n+1}|X_n, U_{n-1}) = (a_n \tilde{x}_n + b_n \tilde{u}_n^*)^T A_{n+1} (a_n \tilde{x}_n + b_n \tilde{u}_n^*)
\]

\[
+ (a_n \tilde{x}_n + b_n \tilde{u}_n^*)^T (A_{n+1} + A_{n+1}^T) c_n Q^n \tilde{r}_n + E(\tilde{v}_n^T c_{n+1} \tilde{v}_m | X_n, U_{n-1})
\]

\[
+ 2 \tilde{r}_n^T (e + Q^n) B_{n+1} (a_n \tilde{x}_n + b_n \tilde{u}_n^*) + \tilde{r}_n^T B_{n+1} c_n Q^n \tilde{r}_n
\]

\[
+ E(\tilde{v}_n^T B_{n+1} c_n \tilde{v}_m | X_n, U_{n-1}) + \tilde{r}_n^T C_{n+1} \tilde{r}_n + \tilde{r}_n^T (C_{n+1} + C_{n+1}^T) Q^n \tilde{r}_n
\]

\[
+ E(\tilde{v}_n^T C_{n+1} \tilde{v}_m | X_n, U_{n-1}) + D_{n+1}^T (e + Q^n) \tilde{r}_n + E_n + 1.
\]

The conditional expectation of \( \tilde{v}_n^T c_{n+1} \tilde{v}_n, \tilde{v}_n^T B_{n+1} c_n \tilde{v}_m, \tilde{v}_n^T C_{n+1} \tilde{v}_n \) when \( X_n, U_{n-1} \) are given, can be calculated analogously to (13)

\[
E(\tilde{v}_n^T F \tilde{v}_m | X_n, U_{n-1}) = \tilde{r}_n^T [(Q^n)^T F Q^n + (Q^n - (Q^n)^2) \text{diag } F] \tilde{r}_n
\]

\[
+ (Q_n^2)^T \text{diag } F \tilde{r}_n + e^T \text{diag } F Q_n^2,
\]

where \( F \) is equal to \( c_{n+1}^T A_{n+1} c_n, B_{n+1} c_n, C_{n+1} \), respectively.

On the other hand from (12) we get

\[
W_n = \tilde{u}_n^* k_n \tilde{u}_n^* + E(\tilde{v}_n^T s_n \tilde{v}_m | X_n, U_{n-1}) + \frac{\varphi_{n+1}}{\varphi_n} E(W_{n+1}|X_n, U_{n-1}).
\]
Using (13), (18), (19) and (20), we obtain from (21) that $W_n$ has the from (15) with:

$$A_n = \frac{\varphi_{n+1}}{\varphi_n} a_n^T A_{n+1} (a_n - b_n \xi_n) + s_n^1,$$

$$B_n = \frac{\varphi_{n+1}}{\varphi_n} [Q^n c_n^T A_{n+1} + (e + Q^n) B_{n+1}] (a_n - b_n \xi_n) + T^n s_n^3,$$

$$C_n = \frac{\varphi_{n+1}}{\varphi_n} \left[ \eta_n^T b_n^T A_{n+1} b_n \eta_n - 2\eta_n^T b_n^T A_{n+1} c_n Q^n - 2(e + Q^n) B_{n+1} b_n \eta_n ight. \right.$$  

$$+ 2B_{n+1} c_n Q^n + C_{n+1} + (C_{n+1} + C_{n+1}^T) Q^n$$  

$$+ Q^n (c_n^T A_{n+1} c_n + 2C_{n+1} c_n + C_{n+1})$$  

$$+ (Q^T - (Q^n)^2) \text{diag}(c_n^T A_{n+1} c_n + 2B_{n+1} c_n + C_{n+1}) \left] 
$$

$$+ \eta_n^T k_n \eta_n + T^n s_n^3 T^n + (T^n)^2 \text{diag} s_n^2,$$

$$\bar{D}_n^T = \frac{\varphi_{n+1}}{\varphi_n} [\bar{Q}_n^n]^T \text{diag}(c_n^T A_{n+1} c_n + 2B_{n+1} c_n + C_{n+1}) + \bar{D}_{n+1}^T (e + Q^n)$$

$$+ (T^n)^2 \text{diag} s_n^2,$$

$$E_n = \frac{\varphi_{n+1}}{\varphi_n} \left[ \bar{e}^T \text{diag}(c_n^T A_{n+1} c_n + 2B_{n+1} c_n + C_{n+1}) \bar{Q}^n + E_{n+1} \right]$$

$$+ \bar{e}^T \text{diag} s_n^2 T^n,$$

where $\xi_n$, $\eta_n$ are given by (18).

While calculating we take into account the fact that $A_n$ is a symmetrical matrix. Then we obtain

**Theorem**

For the $m$-dimensional linear system (1) with the disturbances belonging to the exponential family with the variance being the quadratic function of the mean (3), dependent on an unknown parameter $\lambda$, having the prior distribution given by (4) and the random, bounded horizon $N$, independent of the disturbances, the Bayes control $\bar{u}_n^*$ is given by (18). The Bayes risk is given by (15), where $A_n$, $B_n$, $C_n$, $\bar{D}_n^T$, $E_n$ are computed from (22) with the boundary conditions (16).

### 4. THE SECOND ORDER SYSTEM

Let us consider the second order linear stochastic system defined by the equation

$$x_{n+1} = \alpha_{2,n} x_n + \alpha_{1,n} x_{n-1} + \delta_n u_n + \gamma_n v_n,$$

where $n = 0, 1, \ldots, N-1$, $x_0, x_{-1}$ are given. 

$$x_{n+1} = \alpha_{2,n} x_n + \alpha_{1,n} x_{n-1} + \delta_n u_n + \gamma_n v_n, \quad n = 0, 1, \ldots, N-1, \quad x_0, x_{-1} \text{ are given}, \quad (23)$$
where $x_n$ is the state of the system; $u_n$ is the control; $v_n$ is the disturbance at the time $n$; $\alpha_{1,n}, \alpha_{2,n}, \delta_n, \gamma_n$ are the given constants. The general assumption is that $v_n$, $n = 0, 1, \ldots$ are the independent, identically distributed random variables, the distribution belong to the exponential family and depend on the unknown parameter $\lambda$.

It is assumed that the cost of the control for the given control policy $U$ (the loss function) is given by

$$L(U, X_N) = \sum_{i=1}^{N} (s_i x_i^2 + 2t_i x_i \lambda + w_i \lambda^2 + k_i u_i^2),$$

(24)

where $s_i w_i - t_i^2 \geq 0$, $k_i \geq 0$, $s_i \geq 0$, $s_M > 0$, $w_i \geq 0$ for each $i$.

In order to solve the Bayes control problem for this system we use results of the previous section.

Let us observe that (23) can be written as follows:

$$\overline{\xi}_{n+1} = \bar{a}_n \overline{\xi}_n + \bar{b}_n \bar{u}_n + \bar{c}_n \bar{v}_n, \quad n = 0, 1, \ldots, N-1$$

where

$$\overline{\xi}_n = \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}, \quad \bar{a}_n = \begin{pmatrix} \alpha_{2,n} & \alpha_{1,n} \\ 1 & 0 \end{pmatrix}, \quad \bar{b}_n = \begin{pmatrix} \delta_n & 0 \\ 0 & 0 \end{pmatrix},$$

$$\bar{c}_n = \begin{pmatrix} \gamma_n \\ 0 \\ 0 \end{pmatrix}, \quad \bar{u}_n = \begin{pmatrix} u_n \\ 0 \end{pmatrix}, \quad \bar{v}_n = \begin{pmatrix} v_n \\ 0 \end{pmatrix} \text{ for } n = 0, 1, \ldots, N-1.$$

The cost of the control (24) can be written in the following way

$$L(U, X_N) = \sum_{i=1}^{N} (\overline{y}_i^T S_i \overline{y}_i + \overline{u}_i^T K_i \overline{u}_i)$$

where

$$S_i = \begin{pmatrix} s_i & 0 & 0 \\ 0 & 0 & 0 \\ t_i & w_i & 0 \end{pmatrix}, \quad K_i = \begin{pmatrix} k_i & 0 \\ 0 & 0 \end{pmatrix}, \quad \overline{y}_i = \begin{pmatrix} \overline{y}_i \\ \overline{y}_{i-1} \end{pmatrix}, \quad \overline{x}_i = \begin{pmatrix} x_i \\ x_{i-1} \\ \lambda \end{pmatrix}.$$

From Theorem of Section 3 it follows that for this case the Bayes control is $u_n^* = 0$ and

$$u_n^* = -\frac{\varphi_{n+1} \delta_n}{\varphi_n} x_n + \alpha_{1,n} A_{n+1} x_{n-1} + \left[\gamma_n Q^n A_{n+1} + (e + Q^n) D_{n+1}\right] r_n$$

$$= d_n x_n + e_n x_{n-1} + f_n r_n$$
and the Bayes risk $W_n$:

$$W_n = A_n x_n^2 + 2B_n x_n x_{n-1} + C_n x_{n-1}^2 + 2D_n x_n r_n + 2E_n x_{n-1} r_n + F_n r_n^2 + G_n r_n + H_n, \quad n = 0, 1, \ldots, M$$

where

$$A_n = s_n + k_n d_n^2 + \frac{\phi_n + 1}{\phi_n} [A_{n+1} (\alpha_{2,n} + \delta_n d_n)^2 + 2B_{n+1} (\alpha_{2,n} + \delta_n d_n) + C_{n+1}],$$

$$B_n = k_n d_n e_n + \frac{\phi_n + 1}{\phi_n} [A_{n+1} (\alpha_{1,n} + \delta_n e_n) (\alpha_{2,n} + \delta_n d_n) + B_{n+1} (\alpha_{1,n} + \delta_n e_n)],$$

$$C_n = k_n e_n^2 + \frac{\phi_n + 1}{\phi_n} A_{n+1} (\alpha_{1,n} + \delta_n e_n)^2,$$

$$D_n = t_n T^n + k_n d_n f_n + \frac{\phi_n + 1}{\phi_n} \{A_{n+1} [f_{n+1} (\alpha_{2,n} + \delta_n d_n) + (1 + Q^n) + E_{n+1} (1 + Q^n) + E_n (1 + Q^n)] + B_{n+1} (\alpha_{2,n} + \delta_n d_n) (1 + Q^n) + E_{n+1} (1 + Q^n)\},$$

$$E_n = k_n e_n f_n + \frac{\phi_n + 1}{\phi_n} \{A_{n+1} [(\alpha_{2,n} + \delta_n e_n) \delta_n f_n + (\gamma_n (\alpha_{1,n} + \delta_n e_n) Q^n + D_{n+1} (\alpha_{1,n} + \delta_n e_n) (1 + Q^n)]\},$$

$$F_n = w_n T^n + k_n f_n^2 + \frac{\phi_n + 1}{\phi_n} \{A_{n+1} (\delta_n f_n + (\gamma_n Q^n + Q^n) + F_{n+1} (1 + 2Q^n + Q^n)\},$$

$$G_n = w_n T^n + \frac{\phi_n + 1}{\phi_n} \{A_{n+1} \gamma_n Q^n + 2D_{n+1} \gamma_n Q^n + G_{n+1} (1 + Q^n)\},$$

$$H_n = w_n T^n + \frac{\phi_n + 1}{\phi_n} \{A_{n+1} \gamma_n Q^n + 2D_{n+1} \gamma_n Q^n + H_{n+1}\}$$

with the boundary conditions: $A_M = s_M$, $B_M = C_M = E_M = 0$, $D_M = t_M T_M$, $F_M = w_M T^n$, $G_M = w_M T^n$, $H_M = w_M T^n$.

**APPENDIX**

The generalized exponential hyperbolic secant distribution (GEHS)

The generalized hyperbolic secant distribution was considered by Harkness and Harkness in [3]. Recently Morris [5] has shown that the GEHS distribution belongs to the exponential family with the variance being the quadratic function of the mean. Here we give a short summary of elementary facts connected with this distribution.
The GEHS distribution has the following density with respect to the Lebesgue measure:

\[ p(v; \lambda, q) = S(v, q)(1 + \lambda^2) - q/2 \exp(v \arctg \lambda), \quad \lambda \in \mathbb{R}, \quad q > 0, \quad v \in \mathbb{R} \]  

(25)

where

\[ S(v, q) = \frac{2^{q-2}}{\pi} B \left( \frac{q}{2}, \frac{q}{2} \right) \left( \frac{v}{2} \right)^{q} \left( \frac{v}{2} + \frac{i}{2} \right)^{-q} \left( \frac{1}{1 + \frac{v^2}{(q + 2k)^2}} \right)^{-1}. \]

For the integer \( q \), \( S(v, q) \) has a more clear form:

\[
S(v, q) = \begin{cases} 
1 & q = 1, \\
\frac{1}{2 \cosh \frac{v}{2}} & q = 2, \\
\frac{1}{2 \sinh \frac{v}{2}} & q = 2l + 1, \quad l \geq 1, \\
\frac{1}{2 [2(l+1)! \cosh \frac{v}{2}]} \prod_{k=0}^{l} [v^2 + (2k+1)^2], & q = 2l, \quad l \geq 2.
\end{cases}
\]

The mean and the second moment of the random variable with the density (25) for a given \( \lambda \) are following:

\[ E_{\lambda} v = q \lambda, \]

\[ E_{\lambda} v^2 = q(q + 1) \lambda^2 + q. \]

The conjugate family of the prior distribution \( \pi \) of \( \lambda \) given by the density \( g(\lambda; \beta, r) \) is as follows:

\[ g(\lambda; \beta, r) = C(\beta, r)(1 + \lambda^2)^{-\beta} r^{\beta - 1} \exp(r \arctg \lambda), \quad \beta > 1, \quad r \in \mathbb{R} \]

where \( C(\beta, r) \) has the form (see [2]):

\[ C(\beta, r) = \frac{2^\beta (\beta + 1)}{\pi} B \left( \frac{\beta + 2}{2} + \frac{r}{2}, \frac{\beta + 2}{2} - \frac{r}{2} \right). \]

We have then

\[ E_{\pi} \lambda = \frac{r}{\beta} \quad \text{and} \quad E_{\pi} \lambda^2 = \frac{r^2}{\beta(\beta - 1)} + \frac{1}{\beta - 1}. \]

Let

\[ h(v) = \int_{-\infty}^{+\infty} p(v; \lambda, q)g(\lambda; \beta, r) d\lambda. \]
be the marginal density of the random variable \( v \). We have (see [10])

\[
    h(v) = \frac{S(v, q) C(\beta, r)}{C(\beta + q, r + v)} \quad \text{for} \quad q > 0, \quad (\beta, r) \in (1, \infty) \times R, \quad v \in R
\]

and then

\[
    Ev = \frac{q}{\beta} r \quad \text{and} \quad Ev^2 = \frac{q(q + 1)}{\beta(\beta - 1)} r^2 + \frac{q(q + \beta)}{\beta - 1}.
\]

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STEROWANIE BAYESOWSKIE WIELOWYMIAROWEGO SYSTEMU STOCHASTYCZNEGO

Rozpatrzono problem optymalnego sterowania \( n \)-wymirowego układu liniowego przy addytywnych zakłóceńach i kwadratowej funkcji kosztów. Założono, że składowe wektora zakłóceń mają rozkłady z rodziny wykładniczej zależne od parametrów, których rozkłady \textit{a priori} są znane. Horyzont sterowania jest zmienną losową niezależną od zakłóceń, ograniczoną, o znanym rozkładzie. Korzystając z metody programowania dynamicznego uzyskano analityczną postać algorytmu sterowania optymalnego w układzie zamkniętym. Podano przykład wykorzystania uzyskanych rezultatów do wyznaczenia sterowania bayesowskiego dla dyskretnych układów drugiego rzędu.
БЕЙЕСОВСКОЕ УПРАВЛЕНИЕ МНОГОМЕРНОЙ СТОХАСТИЧЕСКОЙ СИСТЕМОЙ

В работе рассмотрена проблема оптимального управления \( n \)-размерной линейной системой при аддитивных помехах и квадратичной функции стоимости. Предположено, что составляющие вектора помех имеют распределения из экспоненциального семейства, зависящие от параметров, распределения которых априори известны. Горизонт управления является случайной переменной, независимой от помех, ограниченной, с известным распределением.

Используя метод динамического программирования, получили аналитическую форму алгоритма оптимального управления в замкнутой системе. Дан пример использования полученных результатов при определении бейесовского управления для дискретных систем второго порядка.